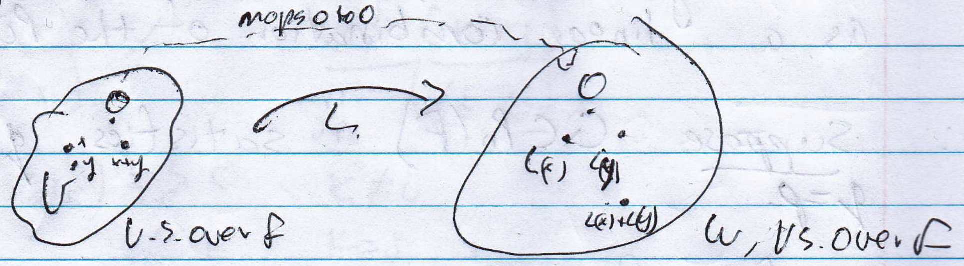
Linear Transformations

Morphism: A function b/w 2 objects of the same kind that respects their structure.



0. $L(0_V) = 0_W$

1. $\forall x, y \in V,\ L(x +_V y) = L(x) +_W L(y)$

2. $\forall x \in V\ a \in F\quad L(ax) = a \cdot L(x)$

Def^n Let V & W be V.S. over some field F. A linear transformation from V to W is a map (function) $L: V \to W$ s.t. 1. & 2. hold.

1. $\forall x, y \in V\quad L(x+y) = L(x) + L(y)$
2. $\forall x \in V, c \in F\quad L(cx) = c\,L(x)$

Claim: If L is a l.t. then $L(0) = 0$

Proof: $L(0_V) = L(0_F \cdot 0_V) \overset{2}{=} 0_F \cdot L(0_V) = 0_W$ □

Claim $L(x-y) = L(x) - L(y)$

Proof: $L(x-y) = L(x+(-1)y)$ Prop. 1

$L(x) + L((-1)y) \overset{2}{=} L(x) + (-1)L(y) = L(x) - L(y)$. $\square$

Claim: $L\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum \alpha_i L(x_i)$

Proof: $L\left(\sum \alpha_i x_i\right) \overset{\text{Prop 1}}{=} \sum L(\alpha_i x_i)$

$\overset{\text{Prop 2}}{=} \sum \alpha_i L(x_i)$ $\square$

Proposition: Given function $L: V \to W$, L is a L.T. iff $\forall x, y, c$ $\quad L(cx+y) = cL(x) + L(y)$.

Proof $\Longrightarrow$ easy.

$\Longleftarrow$ Suppose $L(cx+y) = cL(x) + L(y)$

restricting to $c=1$, i.e. if $c=1$ $\Longrightarrow$ $L(x+y) = L(x) + L(y)$ $\Longrightarrow$ Prop #1

~~restricting y=0~~, i.e. if $y=0, x=0, c=1$ $\Longrightarrow L(0) = L(0) + L(0) \Longrightarrow L(0) = 0$

i.e. if $y=0$ $L(cx) = cL(x) + 0 = cL(x) \Longrightarrow$ Prop # 2