The partition of utility lemma: (Big theorm).
Given a collection $\mathcal{A}$ of open sets in $\mathbb{R}^n$ whose overall union is $A = \bigcup_{U\in\mathcal{A}} U$, there exist a sequence $\{\phi_i\}$ of non-negative compactly supported $\underbrace{}_{C^\infty \text{ functions s.t.}}$ $\text{supp}(\emptyset) = \{x : \phi(x) \neq 0\}$.

Compactly supported: $\text{supp}(\emptyset)$ is compact.
Partitions of Unity Lemma: Given a collection $\mathcal{A}$ of open sets in $\mathbb{R}^k$ whose overall union is $A = \bigcup_{U\in\mathcal{A}} U$. there exists a seq. $\{\phi_i\}$ of non-neg compactly - supp. C^∞ fcns s.t. : 1. $\forall i.\ \exists U \in \mathcal{A}$ s.t. $\text{supp}(\phi_i) \subset U$.
2. Each $\text{supp}\,\phi_i$ is contained in one $U \in A$. 3. $\sum \phi_i(x) = 1$.

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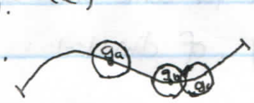
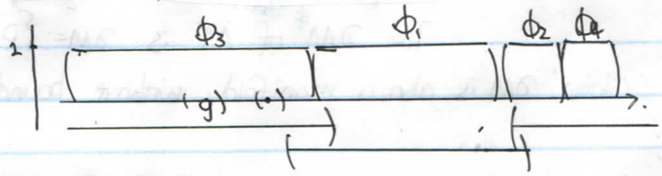
$\partial M = \{\text{bndry pts}\} = \{p \in M : \text{for some patch } \alpha,\ p = \alpha(q) \text{ with } q \in \partial H^k := \mathbb{R}^{k-1} \times \{0\}.\}$.
Thm. With M as before. ∂M is a $(k-1)$-manifold of class C^r with no boundary.
Def. $f : S \xrightarrow{\subset \mathbb{R}^k} \mathbb{R}$ is C^r differentiable if $\forall p \in S$, $\exists$ nbd U of p is $\mathbb{R}^k$ $\exists\, g : U \to \mathbb{R}$ s.t.
1. g is C^r. 2. $g(x) = f(x)$ for $x \in U \cap S$.

Thm.(2). Let S be an arbitrary subset of $\mathbb{R}^k$. A function $f : S \to \mathbb{R}^n$ is said to be of class C^r iff there exists an open sub set U of $\mathbb{R}^k$ containing S and a C^r fcn $g : U \to \mathbb{R}^n$ such that $g(x) = f(x)$ for every $x \in S$.
proof: (of thm 2). $(\Leftarrow)$ ✓
$(\Rightarrow)$ hard.





Partition of units: "division of labor".
Let $\mathcal{A} = \{\underset{\text{open}}{U} : F|_{S\cap U} \text{ has a diff able extension to } U\ \ \exists g : U \to \mathbb{R}^n\ \ 1. g \text{ is } C^r.\ 2. g(x) = f(x) \text{ for } x \in S \cap U\}$
Then $A = \bigcup_{U\in\mathcal{A}} U$ is an open set containing S.
Find a POI. $\{\phi_i\}$ subordimte to $\mathcal{A}$. For each i find $U_i \in \mathcal{A}$ & a C^r: $g_i : U \to \mathbb{R}^n$ s.t.
$g_i|_{U_i \cap S} = f|_{U_i \cap S}$ wrong $g(x) = \sum_{i=1}^{\infty} \phi_i(x) \cdot g(x)$ extended by 0 to A.
g is diff able $g(x) = f(x)$ for $x \in S$. □