13th ~~Fri.~~ Mon. March Hour 062

~~H~~ Read along: 32-34

Claim: If $\phi: \mathbb{R}^n_{x_i} \to \mathbb{R}^n_{y_i}$, and $w = f\,dy_{\underline{n}} \in \Omega^{top}(\mathbb{R}^n_{y_i})$
then, $\phi^*(w) = \det(D\phi)\, \phi^* f\, dx_{\underline{n}} \in \Omega^{top}(\mathbb{R}^n_{x_i})$
$J\phi$ where $dy_{\underline{n}} = dy_1 \wedge \cdots \wedge dy_n$

Pf: $\phi^*(dy_n) = \phi^*(dy_1 \wedge \cdots \wedge dy_n)$, $\phi(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_n(x) \end{pmatrix}$

$= d(\phi^* y_1) \wedge d(\phi^* y_2) \wedge \cdots \wedge d(\phi^* y_n)$

$= (d\phi_1) \wedge \cdots \wedge (d\phi_n)$

$= \left(\frac{\partial \phi_1}{\partial x_1} dx_1 + \frac{\partial \phi_1}{\partial x_2} dx_2 + \frac{\partial \phi_1}{\partial x_3} dx_3 + \cdots + \frac{\partial \phi_1}{\partial x_n} dx_n\right)$

$\wedge \left(\frac{\partial \phi_2}{\partial x_1} dx_1 + \frac{\partial \phi_2}{\partial x_2} dx_2 + \cdots + \frac{\partial \phi_2}{\partial x_n} dx_n\right)$

$\wedge$

$\wedge \left(\frac{\partial \phi_n}{\partial x_1} dx_1 + \frac{\partial \phi_n}{\partial x_2} dx_2 + \cdots + \frac{\partial \phi_n}{\partial x_n} dx_n\right)$

$= \Sigma$ wedge those n terms together picking 1 term from each line $= \Sigma$ (wedge of these terms picking 1 term from each row/col.)

$(dx_i \wedge dx_j = -dx_j \wedge dx_i)$

$$= \sum_{\sigma \in S} \frac{\partial \phi_1}{\partial x_{\sigma_1}} dx_{\sigma_1} \wedge \frac{\partial \phi_2}{\partial x_{\sigma_2}} dx_{\sigma_2} \wedge \cdots$$

$$= \sum_{\sigma} (-1)^{\sigma} \prod_{i=1}^{n} \frac{\partial \phi_i}{\partial x_{\sigma_i}} \Big) dx_{\underline{n}}$$

where σ_i = the sequence # of term in line i

If (A_{ij}) is in $M_{n \times n}(\mathbb{R})$, then $\det A = \sum_{J} (-1)^{J} \prod_{i=1}^{n} a_{i,\sigma_i}$

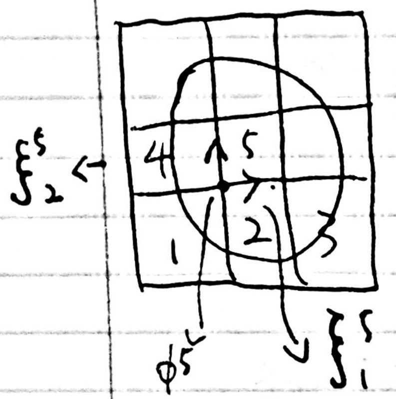
So # $= \det \left(\frac{\partial \phi_i}{\partial x_j}\right) dx_{\underline{n}} = \det(\phi)\, dy_{\underline{n}}$. Q.

More generally, if ~~w~~ $w=\sum_{I\in\binom{n}{k}} a_I\, dy_I \in \Omega^k(\mathbb{R}^n y_i)$.

then $w = \sum_{I\in\binom{n}{k}}\sum_{J\in\binom{n}{k}} (\phi^* a_I \det(D\phi)_{I,J})\, dx_J$.

where $(D\phi)_{I,J}$ stands for the $k\times k$ matrix obtained by considering only rows J and columns I of $D\phi$

$w\in\Omega^{top}(\mathbb{R}^k x_i)$, $\mathrm{Supp}(w) = \{$set of $x\in\mathbb{R}^k$ such that $w(x)\neq 0\} \subset Q$ a rectangle

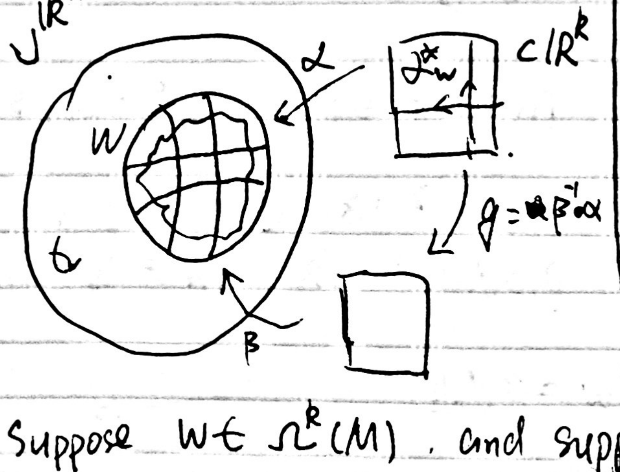


$w = F\, dx_{\underline{n}}$; intuitively $\int_Q w \doteq \sim \sum$, n is big

$\int_Q w = \int_{\mathbb{R}^k} w := \int_{\mathbb{R}^k} F = \int_Q F$ take w, evaluating on rectangle and sum them

$$\sum_{i=1}^N w(\xi_1^i, \xi_2^i, \dots) = \sum_{i=1}^N (F(x_i)\cdot(dx_1\wedge dx_2)(\xi_1^i,\xi_2^i))$$
$$= \sum_{i=1}^N (F(p_i)\cdot|\xi_1^i|\cdot|\xi_2^i|)$$
$$= \sum_i F(p_i)\cdot \mathrm{Vol}(R_i) \approx \int_Q F$$

n is big



Suppose $w\in\Omega^k(M)$, and supp $w\subset U = \mathrm{im}(\alpha)$

$$\int_M w = \int_A \alpha^* w \overset{?}{=} \int_B \beta^* w \quad \text{where } \alpha = \beta\circ g$$

preliminary; $A \subset \mathbb{R}^k$; depend on α?; $B\subset\mathbb{R}^k$

So $\int_A\alpha^* w = \int_A (\beta\circ g)^* w = \int_A g^*\circ \underbrace{\beta^* w}_{top} = \cdots = \pm\int_B \beta^* w$