November 10^{th}, 2009

MAT240: Abstract Linear Algebra Lecture:

Let $A \in M_{mxn}$ be a matrix. Let T_A be the linear transformation $T_A: F^n \to F^m$ given by:

$$v \in F^n = M_{nx\,1} \to A * v \in M_{mx\,1} = F^m$$

The Good and the Bad of Matrix Multiplication:

Good	**Bad**
1. $A + B = B + A$ $A + (B + C) = (A + B) + C$ M_{mxn} is a V.S. $L(V, W)$ is a V.S.	1. $A + B$ is defined only if dimensions match
2. $(AB)C = A(BC)$ (when it makes sense) $\leftrightarrow$ $(T_A\ compose\ T_B)\ compose\ T_C$ $= T_A\ compose\ T_B\ (compose\ T_C)$ Composition is always associative!	2. AB is defined only when: $(\#\ columns\ of\ A) = (\#\ rows\ of\ B)$
3. $\forall n \exists I_n \in M_{mxn}$ s.t. $A * I_n = A$, $I_m * A = A$ $A \in M_{mxn}$ Proof: $v \xrightarrow{I} v$ $F^n \xrightarrow{I} F^n$ $(I)_{ei}^{ei} = ([I_{(ei)}] \quad \cdots) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & \vdots \\ \vdots & 0 & 1 \end{pmatrix}$	3. $A \neq 0\ does\ not\ imply\ \exists\ A^{-1}$ $AB \neq BA$
4. If $A \in M_{mxn}$ & $A^{-1}\ s.t.\ A * A^{-1} = I$ then $A^{-1}A = I$	
5. $A(B + C) = AB + AC$ $(A + B)C = AC + BC$ Proof 1: Work in M (exercise) Proof 2: Work in l.t. (exercise)	

Goals:

1. Given $T: V \to W$ compute rank T
2. Given an nxn matrix compute A^{-1} (if $A^{-1}\exists$)
3. Solve large systems of linear equations systematically.

Proposition:

$$V` \xrightarrow{Q} V \xrightarrow{T} W \xrightarrow{P} W`$$

$$V` \xrightarrow{T`} W`$$

Given this diagram, if P & Q are invertible, then rank(T) = rank(PTQ)

Proof:

$P: R(T) \rightarrow R(T`)$ is an isomorphism.

1. $P: W \rightarrow W`$ if $w \in R(T)$ then $P: W \in R(T`)$
2. P is onto.
3. P is one-to-one.

Proof of 1:

If $w \in R(T)$ then find $v \in V$ s.t. $Tv = w$

Let $v = Q^{-1}v \in V`$

$T`v` = (PTQ)(v`) = PTQQ`v$

$= PTv = PW$

So $PW \in R(T`)$■

Proof of 2:

If $w` \in R(T`)$ find $v` \in V`$ s.t. $T`v` = w`$

but then $w`$ is $T`v` = PTQv` = P\left(T(Qv`)\right) = P(T(v)) = P(w)$■

Proof of 3:

Automatic! (Restricting a one to one function is always one-to-one!)

<u>Definition:</u>

If $A \in M_{mxn}$ let $T_A: F^n \rightarrow F^m$ be the usual, $T_A v = Av$ and set:

$$rank(A) = rank(T)$$

Comment 1: Given $\underset{\beta}{V} \overset{T}{\rightarrow} \underset{\gamma}{W}$

$rank[T_A]_\beta^\gamma = rank(T)$

Proof: Let $A = [T_A]_\beta^\gamma$, we have:

$$\begin{array}{ccc} \boldsymbol{V} & \overrightarrow{T} & \boldsymbol{W} \\ \downarrow Q & (system\ is\ commutative) & \downarrow P \\ \boldsymbol{F^n} & \underset{T_A}{\rightarrow} & \boldsymbol{F^m} \end{array} \qquad \rightarrow T_A = PTQ^{-1}$$

$rank[T_A]_\beta^\gamma = rank(A) = rank(T_A) = rank(PTQ^{-1}) = rank(T)$ ■

Comment 2: If $A \in M_{mxn}(F), P \in M_{mxm}, Q \in M_{nxn}$ & P, Q are invertible, then:

$$rank(A) = rank(PAQ)$$

Proof:

(Exercise)