Problem Set 15 — MAT257

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Problems marked with $*$ are to be submitted for credit.

1 Munkres §29 (p.251)

$*$ 1. Let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be $\mathcal{C}^r$. Show that the velocity vector of γ corresponding to the parameter value t is the vector $\gamma_*(t; \mathbf{e}_1)$.

$*$ 2. If A is open in $\mathbb{R}^k$ and $\alpha : A \to \mathbb{R}^n$ is $\mathcal{C}^r$, show that $\alpha_*(\mathbf{x}; \mathbf{v})$ is the velocity vector of the curve $\gamma(t) = \alpha(\mathbf{x} + t\mathbf{v})$ corresponding to the parameter value $t = 0$.

3. Let M be a $\mathcal{C}^r$ k-manifold in $\mathbb{R}^n$. Let $\mathbf{p} \in M$. Show that the tangent space to M at $\mathbf{p}$ is well-defined, independent of the choice of coordinate patch.

$*$ 4. Let M be a $\mathcal{C}^r$ k-manifold in $\mathbb{R}^n$. Let $\mathbf{p} \in M \setminus \partial M$.

   (a) Show that if $(\mathbf{p}; \mathbf{v})$ is a tangent vector to M, then there is a parametrized curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ whose image set lies in M, such that $(\mathbf{p}; \mathbf{v})$ equals the velocity vector of γ corresponding to the parameter value $t = 0$.

   (b) Prove the converse.

5. Let M be a $\mathcal{C}^r$ k-manifold in $\mathbb{R}^n$. Let $\mathbf{q} \in \partial M$.

   (a) Show that if $(\mathbf{q}; \mathbf{v})$ is a tangent vector to M at $\mathbf{q}$, then there is a parametrized curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$, where γ carries either $(-\epsilon, 0]$ or $[0, \epsilon)$ into M, such that $(\mathbf{q}; \mathbf{v})$ equals the velocity vector of γ corresponding to the parameter value $t = 0$.

   (b) Prove the converse.

2 Munkres §30 (pp.260–262)

1. Let A be open in $\mathbb{R}^n$.

   (a) Show that $\Omega^k(A)$ is a vector space.

   (b) Show that the set of all $\mathcal{C}^\infty$ vector fields on A is a vector space.

2. Consider the forms

$$\omega = xy\,dx + 3\,dy - yz\,dz$$
$$\eta = x\,dx - yz^2\,dy + 2x\,dz,$$

in $\mathbb{R}^3$. Verify by direct computation that

$$d(d\omega) = 0,$$
$$d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge d\eta.$$

3. Let ω be a k-form defined in an open set A of $\mathbb{R}^n$. We say that ω vanishes at $\mathbf{x}$ if $\omega(\mathbf{x})$ is the zero tensor.
   (a) Show that if ω vanishes at each $\mathbf{x}$ in a neighbourhood of $\mathbf{x}_0$, then $d\omega$ vanishes at $\mathbf{x}_0$.
   (b) Give an example to show that if ω vanishes at $\mathbf{x}_0$, then $d\omega$ need not vanish at $\mathbf{x}_0$.
4. Let $A = \mathbb{R}^2 \setminus \{\mathbf{0}\}$; consider the 1-form in A defined by the equation

$$\omega = \frac{x\,dx + y\,dy}{x^2 + y^2}.$$

   (a) Show that ω is closed.
   (b) Show that ω is exact on A.
5. Prove the following:

Theorem. Let $A = \mathbb{R}^2 \setminus \{\mathbf{0}\}$; let

$$\omega = \frac{-y\,dx + x\,dy}{x^2 + y^2}$$

in A. Then ω is closed, but not exact, in A.

Proof.

   (a) Show that ω is closed.
   (b) Let B consist of $\mathbb{R}^2$ with the non-negative x-axis deleted. Show that for each $(x, y) \in B$, there is a unique t with $0 < t < 2\pi$ such that

$$x = \sqrt{x^2 + y^2} \cdot \cos t$$
$$y = \sqrt{x^2 + y^2} \cdot \sin t;$$

   denote this value of t by $\phi(x, y)$.
   (c) Show that ϕ is of class $\mathcal{C}^\infty$. [*Hint:* The inverse sine and inverse cosine functions are $\mathcal{C}^\infty$ on the interval $(-1, 1)$.]
   (d) Show that $\omega = d\phi$ in B. [*Hint:* We have $\tan\phi = y/x$ if $x \neq 0$ and $\cot\phi = x/y$ if $y \neq 0$.]
   (e) Show that if g is a closed 0-form in B, then g is constant in B. [*Hint:* Use the mean-value theorem to show that if $\mathbf{a}$ is the point $(-1, 0)$ of $\mathbb{R}^2$, then $g(\mathbf{x}) = g(\mathbf{a})$ for all $\mathbf{x} \in B$.]
   (f) Show that ω is not exact in A. [*Hint:* If $\omega = df$ in A, then $f - \phi$ is constant in B. Evaluate the limit of $f(1, y)$ as y approaches 0 through the positive and negative values.]
6. Let $A = \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Let m be a fixed positive integer. Consider the following $n - 1$ form in A:

$$\eta = \sum_{i=1}^{n} (-1)^{i-1} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

where $f_i(\mathbf{x}) = x_i / \|\mathbf{x}\|^m$, and where $\widehat{dx_i}$ means that the factor dx_i is to be omitted.

(a) Calculate $d\eta$.

(b) For what values of m is it true that $d\eta = 0$?

7. Prove the following, which expresses d as a generalized "directional derivative":

Theorem. Let A be open in $\mathbb{R}^n$; let ω be a $k-1$ form in A. Given $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$, define

$$h(\mathbf{x}) = d\omega(\mathbf{x})\big((\mathbf{x};\mathbf{v}_1), \ldots, (\mathbf{x};\mathbf{v}_k)\big),$$
$$g_j(\mathbf{x}) = d\omega(\mathbf{x})\big((\mathbf{x};\mathbf{v}_1), \ldots, \widehat{(\mathbf{x};\mathbf{v}_j)}, \ldots, (\mathbf{x};\mathbf{v}_k)\big),$$

where $\widehat{a}$ means that the component a is to be omitted. Then

$$h(\mathbf{x}) = \sum_{j=1}^{k} (-1)^{j-1} Dg_j(\mathbf{x}) \cdot \mathbf{v}_j.$$

Proof.

(a) Let $X = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix}$. For each j, let $Y_j = \begin{bmatrix} \mathbf{v}_1 & \cdots & \widehat{\mathbf{v}_j} & \cdots & \mathbf{v}_k \end{bmatrix}$. Given $(i, i_1, \ldots, i_{k-1})$, show that

$$\det X(i, i_1, \ldots, i_{k-1}) = \sum_{j=1}^{k} (-1)^{j-1} v_{ij} \det Y_j(i, i_1, \ldots, i_{k-1}).$$

(b) Verify the theorem in the case $\omega = f\, dx_I$.

(c) Complete the proof.