$(A^T)_{ij} = A_{ji}$ $R^T = \alpha$

$(\sim)$

${E^1_{ij}}^T = E^1_{ij}$

${E^2_{i,c}}^T = E^2_{i,c}$

${E^3_{ij,c}}^T = E^3_{j,i,c}$

$\Rightarrow \det E^T = \det E$

for any elem E

Lemma $(A \cdot B)^T = B^T A^T$

PE miserable pf:

$((AB)^T)_{ki} = (AB)_{ik} = \sum_j A_{ij} B_{jk}$

$(B^T A^T)_{ki} = \sum_j (B^T)_{kj} (A^T)_{ji} = \sum_j B_{jk} \cdot A_{ij}$

Fun PF

mirror

$$\begin{matrix} & (B\,I) \\ (A) & (A.B) \end{matrix} \rightsquigarrow \begin{matrix} & (A^T) \\ (B^T) & ((AB)^T) \end{matrix}$$

$$\Rightarrow B^T A^T = (AB)^T \qquad \square$$

Thm 4 If $A \in M_{n\times n}$, then
$\det(A^T) = \det(A)$

PF If A is invertible $A = E_1 \cdots E_k$
& then

$A^T = (E_1(E_2 \cdots E_k))^T = (E_2 \cdots E_k)^T E_1^T$

$= E_k^T \cdots E_2^T E_1^T$

So $\det(A^T) = \det(E_k^T E_{k-1}^T \cdots E_1^T)$

$= |E_k^T| \cdot |E_{k-1}^T| \cdots |E_1^T|$

$= |E_1| \cdot |E_2| \cdots |E_k^T|$

$= \det(E_1 \cdot E_2 \cdots E_k) = \det(A)$

If A isn't invertible,

$n > \text{rank} A = \text{rank} A^T$

$\Rightarrow A^T$ isn't invertible.

$\det(A) = 0 = \det(A^T)$

Thm 5 Everything that's for rows, is also true for cols

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} = (-2) \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}$$

$$= (-2) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = -2$$

skipped extras

1. other formulas for det.
2. cofactors along first col
3. cofactor formular using row # i or col # j