Solutions to Homework 1

12.2 Let R={0,2,4,6,8}, +, x modulo 10
By Theorem 12.2, if R has a unity it is unique. Verification that 6 is that unity is computational.
6x0=0, 6x2=12=2 mod 10, 6x4=24=4 mod 10, 6x6=36=6 mod 10, 6x8=48=8 mod 10
Therefore, 6xa=a mod 10 and 6 is the unity element of R.
Alternately, if a is the unity, then $a^2=a$ in R. The only element of the set with the property that $a^2=a$ mod10 is 6 and therefore if R has a unity it must be 6.

12.13 Let R=Z under the usual operations and let S be a nontrivial (not {0}) subring of R.
Since S is a subring and S ≠ {0}, ∃ a ∈ S a≠0.
Since S is a subring, it is closed under subtraction and therefore 0-a = -a ∈ S.
Therefore there exists some positive integer a ∈ S.
Therefore, by the well-ordering principle for natural numbers, there exists some smallest positive integer, $a_0 \in S$.
Since a_0 e S, $-a_0 \in S$, therefore $2a_0$ e S and similarly ka_0 e S for all k e Z by $a_0-(-a_0)-(-a_0)\ldots-(-a_0)=ka_0$.
Therefore for all subrings S of R, a_0Z is contained within S for the smallest element a_0 in S.
Assume there is some element b in S that is not equal to any ka_0, k ∈ Z.
Then $ma_0<b<(m+1)a_0$ for some m ∈ Z.
Therefore $0<b-ma_0<a_0$ and $b-ma_0$ e S (closed under subtraction and both are in S).
But, we assumed that a_0 was the smallest element in S; a contradiction.
Therefore, for all b ∈ S, $b=ra_0$ for some r e Z, a_0 the smallest positive element in S.
Therefore, for all subrings S of Z, S=kZ, k ∈ N, or S={0}.

12. 19 Let R be a ring and let Z(R)={x e R | ax=xa for all a ∈ R}. Using the operations from R:
Z(R) is non empty: 0 ∈ Z(R) since 0a=0=a0 for all a ∈ R.
Let x,y ∈ Z(R):

> Then, for all a ∈ R: (xy)a=x(ya)=x(ay)=(xa)y=(ax)y=a(xy) by the Associative Law followed by the property of elements of Z(R) repeatedly.
> Therefore, xy ∈ Z(R) whenever x,y ∈ Z(R)
>
> Also, for all a ∈ R: (x-y)a=xa-ya=ax-ay=a(x-y) by the Distributive Law and the property of elements of Z(R).
> Therefore, x-y ∈ Z(R) whenever x,y ∈ Z(R).

Therefore Z(R) is closed under multiplication and subtraction and by Theorem 12.3 the center of a ring is a subring.

12.22 Let R be a commutative ring with unity. Then let U(R)={u ∈ R | ∃ u^{-1} ∈ R}.
To check the axioms of groups (under multiplication from R):

1. Associativity: Let $u_1,u_2,u_3 \in U(R)$: Then in particular u_1,u_2,u_3 are in R and since multiplication in R is associative, it is associative in U(R).
2. Invertibility: Let $u_1 \in U(R)$: Then there exists a $u_1^{-1} \in R$.
   Therefore $u_1u_1^{-1}=e=u_1^{-1}u_1$ where e is the unity element of R.
   Therefore $u_1=(u_1^{-1})^{-1}$
   Therefore $u_1^{-1} \in U(R)$ whenever $u_1 \in R$.
3. Identity: R has a unity. Let this unity be denoted by e (from above):
   $e^{-1}e=e=ee^{-1}$ if e^{-1} exists. But ee=e and therefore $e=e^{-1}$ and therefore e ∈ U(R).

Also, to show that it is closed under the binary operation:
If a,b ∈ U(R), then a^{-1},b^{-1} exist in R:

Therefore $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}=aea^{-1}=aa^{-1}=e$ and R is commutative so the same holds on the left.
Therefore U(R) is closed ($a,b \in U(R)$ implies that $ab \in U(R)$).
Therefore U(R) is a group under multiplication from R.

13.13 Let R be a ring with unity and let a be nilpotent.
Then there exists $n \in N$ such that $a^n=0$.
Since addition is associative and commutative, the expression:
$1+a+a^2+a^3+\ldots+a^{n-1}$ is permissible.
Denote this polynomial in a as b.
$b \in R$ since R is closed under multiplication and addition and 1 e R.
Therefore $(1-a)b=1b-ab=b-ab = (1+a+a^2+\ldots+a^{n-1}) - (a+a^2+a^3+\ldots+a^n)$ and by associativity and commutativity this equals: $1-a^n = 1$ since an=0.
Similarly we can reverse this and show that $b(1-a)=1$ and therefore that $b=(1-a)^{-1}$ and 1-a has a multiplicative inverse in R.

13.24 Let d e N, $R=Q[\sqrt{(d)}]=\{a+b\sqrt{(d)} \mid a,b \in Q\}$ under the operations from the Reals.
Checking the axioms to show that this is a ring is trivial:
Since R is contained within **R** and borrows its operations from it, and because the reals satisfy commutativity and associativity of addition as well as associativity of multiplication and distributivity, therefore R satisfies these axioms as well.
Axiom 3: 0 e R: $0=0+0\sqrt{(d)}$ and 0 e Q and to check that 0+a=a is trivial.
Axiom 4: Additive inverses: For any $x=a+b\sqrt{(d)}$ let $-x=-a-b\sqrt{(d)}$ and show that their sum is 0.
Therefore R is a ring.
Since R is in **R** and **R** is a field, multiplication is commutative in R and also R borrows its unity from **R.**

To show that every element has a multiplicative inverse:
If $d=k^2$ for some $k \in N$ then R=Q and for all $x \in Q$, $x\neq 0$, $1/x=x^{-1}$.
If $d\neq k^2$ for any $k \in N$, then if $x \in R$, x nonzero, $x=m/n + p\sqrt{(d)}/q$ with neither n nor q zero and not both m and p zero.
Then if $m^2/n^2 - p^2d/q^2 =0$, then $m^2q^2=p^2n^2d$ and $|mq|=|pn|\sqrt{(d)}$.
But the left side is rational and the right side irrational.
Therefore, if $x=a+b\sqrt{(d)}$ and $d\neq k^2$ for some $k \in N$, then $a^2-b^2d\neq 0$.
Therefore solve for 1/x by multiplying by the conjugate; yielding:
$1/x=a-b\sqrt{(d)}[1/(a^2-b^2d)]$ and recall that the denominator is nonzero. By properties of the reals 1/x is x^{-1}.
Therefore R is a commutative ring with unity in which every nonzero element is a unit.
Therefore R is a field.