Jan 16

Thm. Given $A \xrightarrow[\text{1-1, onto}]{g} B \xrightarrow{f} \mathbb{R}$, $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^n$. $\int_B f = \int_A (f\circ g)|\det Dg|$

Corollary: Let $P(v_1 \cdots v_n)$ be the parallelpiped spanned by $v_1 \cdots v_n$

$P(v_1 \cdots v_n) = \{\sum a_i v_i \,;\, 0 \le a_i \le 1\}$ Then $\operatorname{vol}(P(v_1 \cdots v_n)) = |\det(v_1|v_2|\cdots|v_n)|$

"geometric interpretation of det"

pf: $\underset{[0,1]^n}{A} \xrightarrow{g} \underset{P(v_1\cdots v_n)}{B} \xrightarrow{f=1} \mathbb{R}$. $\quad g\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix} \longmapsto \sum x_i v_i = M\begin{pmatrix}x_1\\ \vdots \\ x_n\end{pmatrix}$

Clearly $P(v_1 \cdots v_n) = g([0,1]^n)$ where $M = (v_1|v_2|v_3|\cdots|v_n)$

By CVT $\int_B 1 = \int_{P(v_1\cdots v_n)} 1 = \operatorname{vol}(P(v_1 \cdots v_n))$

$\overset{\text{CVT}}{\|}$

$\int_A 1\,|\det Dg| = \int_{[0,1]^n} |\det(v_1|\cdots|v_n)| = 1 \cdot |\det(v_1|\cdots|v_n)|$ $\square$.

Exercises. Compute the volumn of $B^3 = \{x \in \mathbb{R}^3 : \|x\| < 1\}$. $\|x\| = (\sum x_i^2)^{1/2} = (x^T x)^{1/2}$

$g: [0,1]\times[-\pi,\pi]\times[-\frac{\pi}{2},\frac{\pi}{2}] \longrightarrow B^3 \subset \mathbb{R}^3_{x,y,z}$ where $g\begin{pmatrix}r\\ \theta\\ \varphi\end{pmatrix} = \begin{pmatrix} r\cos\varphi\cos\theta \\ r\cos\varphi\sin\theta \\ r\sin\varphi\end{pmatrix}$

$\|g(r,\theta,\varphi)\|^2 = r^2\cos^2\varphi\cos^2\theta + r^2\cos^2\varphi\sin^2\theta + r^2\sin^2\varphi = r^2$

$Dg = \begin{pmatrix} \cos\phi\cos\theta & -r\cos\phi\sin\theta & -r\sin\phi\cos\theta \\ \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi & 0 & r\cos\phi \end{pmatrix}$ $\quad |\det(Dg)| = \sin\phi = |r^2\cos\phi|$

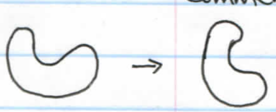
$\operatorname{Vol}(B^3) = \int_{B^3} 1 \overset{\text{CVT}}{=} \int_{[0,1]\times[-\pi,\pi]\times[-\frac{\pi}{2},\frac{\pi}{2}]} r^2\cos\phi \overset{\text{Fub}}{=} \int_0^1 dr \int_{-\pi}^{\pi} d\theta \int_{-\pi/2}^{\pi/2} r^2\cos\varphi\, d\varphi$

$= \int_0^1 dr \int_{-\pi}^{\pi} d\theta\, 2r^2 = \int_0^1 dr\, 4\pi r^2 = \left[\frac{4}{3}\pi r^3\right]_{-1}^{1} = \frac{8}{3}\pi$.

Def. $h: \mathbb{R}^n \to \mathbb{R}^n$ is called an "isometry" if $\forall x,y$, $d(h(x),h(y)) = d(x,y)$

Thm. $h: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry iff it can be written in the form $h(x) = P + Ax$ when $P \in \mathbb{R}^n$ and $A \in M_{n\times n}$ s.t. $A^T \cdot A = Id$.

Comments: 1. isometries are "volume preserving"



pf: By CVT, ... $|\det(Dh)| = |\det(A)| = |\pm 1| = 1$

$\det(Id) = \det(A^T A) = \det(A^T)\det(A) = \det(A)^2 \Rightarrow \det(A) = \pm 1$