MAT 240.

Goal for next few weeks:

$\boxed{\text{All vector spaces are the "same" as } F^n}$

$$F^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \right\}$$

1. What's "same"?
   ↳ b/c can id members in each set w/ e/o.
2. So what?
   ↳ b/c may be useful to id objects w/ #'s.
   ↳
3. So why bother w/ vector spaces?
   ↳ any vectorspace can boil down to working w/ matrices.
4. How is it proven?

Consider
$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly every element $v \in F^n$ can be written in a unique way as a linear comb. of those.

eg
$$\begin{bmatrix} 2 \\ 7 \\ 8 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We will show that every V.S. has such a "basis." - a collection of vectors with which you can write every other vector in a unique way.

Def: $v \in V$ is a linear combination of elements in $S \in V$ if $\exists\, u_1, \dots u_n \in S$ & $a_1 \dots a_n \in F$
s.t. $v = \sum a_i u_i$

Eg. In $P_3(\mathbb{R})$ = (Polynomials of degree ≤ 3 w/ Coefficients in $\mathbb{R}$),
$v_1 = 2x^3 - 2x^2 + 12x - 6$
is a linear combination of
$u_1 = x^3 - 2x^2 - 5x - 3$ & $u_2 = 3x^3 - 5x^2 - 4x - 9$
but $v_2 = 3x^3 - 2x^2 + 7x + 8$
is not.

Why?

$$\begin{cases} 2x^3 - 2x^2 + 12x - 6 = v_1 \overset{?}{=} a_1 u_1 + a_2 u_2 \\ \qquad = a_1(x^3 - 2x^2 - 5x - 3) + a_2(3x^3 - 5x^2 - 4x - 9) \end{cases}$$

b/w #S:

$$\left\{ \begin{array}{lll} \text{Equal coefficients for } x^3: & 2 = a_1 + 3a_2 & a_1 = -4 \\ \quad " \qquad " \qquad " \quad x^2: & -2 = -2a_1 - 5a_2 & a_2 = 2 \\ \quad " \qquad " \qquad " \quad x^1: & 12 = -5a_1 - 4a_2 & \\ \quad " \qquad " \qquad " \quad x^0: & -6 = -3a_1 - 9a_2 & \end{array} \right.$$

$2(1) - (2)$ $\quad 2 = a_2$
(1) $\quad 2 = a_1 + 3 \cdot 2 \Rightarrow a_1 = -4$

Now, check that vals satisfy all 4 eq'ns. Yes, they do.
$\therefore$ $a_1 = -4$ & $a_2 = 2$ solve bottom & top systems.

$\therefore$ $v_1 = -4u_1 + 2u_2$ & we're done.

MAT240.

Check that v_2 isn't a l.c. of u_1 & u_2:

Equal coefficient for x^3: $3 = a_1 + 3a_2$

" " " x^2: $-2 = -2a_1 - 5a_2$

" " " x^1: $7 = -5a_1 - 4a_2$

" " " x^0: $8 = -3a_1 - 9a_2$

$2①+②$ $\quad 4 = a_2$ $\quad \boxed{a_2 = 4}$

$①$ $\quad 3 = a_1 + 3 \cdot 4 \Rightarrow \boxed{a_1 = -9}$

these vals solve ① & ②

Now check ③ & ④

$7 = -5(-9) - 4 \cdot 4$ No. $\therefore$ These vals aren't a

$\therefore$ there isn't a sol'n

$\therefore$ v_2 isn't l.c.

Aside: eg Show that $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^2$ is a lin. comb. of $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $u_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Indeed, $v_1 = 2u_1 + 3u_2 + 0u_3$

$= 0u_1 + 1u_2 + 2u_3$ $\quad$ So, basis is not unique.

Def. We say that a subset $S \subset V$ "generates" or "spans" V, if ~~span of subset~~ span $S = \{$all l.c. of elements in $S\} =$

Eg. $V = M_{2\times 2}(\mathbb{R})$ $\quad M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$N_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $N_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $N_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $N_4 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Claims: ① $\{M_1, M_2, M_3, M_4\}$ generates V

② $\{N_1, N_2, N_3, N_4\}$ Generates V.

③ $\{M_1, M_2, M_3\}$ doesn't generate V.

④ $\{N_1, N_2, N_3\}$ " " "

Pf ①: Given any $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, need to find $a_1 \dots a_4$ s.t. $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = B = a_1 M_1 + a_2 M_2 + a_3 M_3 + a_4 M_4$.

$$= \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a_4 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$$

$$\Leftrightarrow \left.\begin{array}{l} b_{11} = a_1 \\ b_{12} = a_2 \\ b_{21} = a_3 \\ b_{22} = a_4 \end{array}\right\}$$ System of 4 equtns w/ 4 unknowns. $(a_1, \dots a_4)$

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = B = a_1 N_1 + a_2 N_2 + a_3 N_3 + a_4 N_4$$

$$= \begin{bmatrix} 0 & a_1 \\ a_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 \\ a_2 & a_2 \end{bmatrix} + \begin{bmatrix} a_3 & a_3 \\ 0 & a_3 \end{bmatrix} + \begin{bmatrix} a_4 & a_4 \\ a_4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} a_2 + a_3 + a_4 & a_1 + a_3 + a_4 \\ a_1 + a_2 + a_4 & a_1 + a_2 + a_3 \end{bmatrix}$$

$$\Leftrightarrow \begin{array}{l} b_{11} = a_2 + a_3 + a_4 \\ b_{12} = a_1 + a_3 \; a_4 \\ b_{21} = \quad \vdots \\ b_{22} = \quad \vdots \end{array}$$

Trick: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $M_1 = N_1 + N_2 + N_3 + N_4 = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} - 3N_1 = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$

$M_2 =$ $\qquad \frac{1}{3}\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\vdots$

Have shown that M_1 is l.c. N_s: $\frac{2}{3} N_1 + \frac{1}{3} N_2 + \frac{1}{3} N_3 + \frac{1}{3} N_4$.

etc.

MAT240.

$B = b_{11} M_1 + b_{12} M_2 + b_{21} M_3 + b_{22} M_4$

$= b_{11}\left(-\frac{2}{3}N_1 + \frac{1}{3}N_2 + \ \right) + b_{12}\left(\frac{1}{3}N_1 + \frac{2}{3}N_2 + \ldots\right) + \ldots$

$=$ a l.c. of $N_1 \ldots N_4$.

$\square$.

Pf. 3. Indeed, in $a_1 M_1 + a_2 M_2 + a_3 M_3 = \begin{bmatrix} a_1 & a_2 \\ a_3 & 0 \end{bmatrix}$

lower right is always 0 so $\begin{pmatrix} 240 & 157 \\ e & \pi \end{pmatrix}$ not in span.

Pf 4. $a_1 N_1 + a_2 N_2 + a_3 N_3 = \begin{bmatrix} a_2 + a_3 & a_1 + a_3 \\ a_1 + a_2 & a_1 + a_2 + a_3 \end{bmatrix}$

$\begin{bmatrix} 240 & 157 \\ e & \pi \end{bmatrix} \overset{?}{\neq}$

$240 = a_2 + a_3$
$157 = a_1 + a_3$
$e = a_1 + a_2$
$\pi = a_1 + a_2 + a_3$

$\xRightarrow{\text{Solve}}$ No solution.

To find counterexample, simply try simple examples.

~~Def~~

Motivation: $S \subset V$ is linearly dependent if it is "wasteful";
i.e. if $\exists v \in V$ s.t. $\exists a_1, \dots a_n \in F$ & $u_1, \dots u_n \in S$
& $b_1, \dots b_m \in F$ & $w_1, \dots w_m \in S$
so that
$$\sum_{i=1}^{n} a_i u_i = v = \sum_{i=1}^{m} b_i w_i$$

$$\underbrace{\sum a_i u_i - \sum b_i w_i}_{\boxed{\sum c_i z_i = 0}} = 0$$

Def. $S \subset V$ is called "linearly dependent" if you can find $z_1 \dots z_n \in S$ different from e/o & $c_1 \dots c_n \in F$ ~~[illegible]~~ not all of which are 0, so that $\sum c_i z_i = 0$

Otherwise S is called linearly independent.

Eg 1. In $\mathbb{R}^3$ $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$ is lin. dep.

$$1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-2)\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 1\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = 0.$$

Eg 2 $\mathbb{R}^n$, $e_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ i^{th} row $\quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

$$S = \{e_1, e_2, \dots, e_n\}$$

Claim: S is lin. indep.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad 0 = \sum_{i=1}^{n} a_i e_i = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \Rightarrow \begin{matrix} a_1 = 0 \\ a_2 = 0 \\ \vdots \\ a_n = 0 \end{matrix}$$

But: not not all a_i's are 0.
$\Rightarrow$ " " " "

$\therefore$ proof of lin dep failed
$\therefore$ lin. indep.

MAT240.

<u>Claim</u>: $S \subset V$ is linearly independent iff whenever $\sum a_i u_i = 0$ & $u_i \in S$ distinct then $\forall_i\, a_i = 0$.

distinct.

Utility

<u>Comments</u>. 1. $\emptyset$ - the empty set $\subset V$ is lin. indep.

2. Suppose $u \in V$

$\{u\}$ is linearly independent iff $u \neq 0$.

"Singleton set"

$\{0\}$ is linearly dependent $7 \cdot 0 = 0$.

If $u \neq 0$,

$au = 0$, & $a \neq 0$

$\Rightarrow a^{-1} a u = 0$,

$u = 0$ $\Rightarrow\Leftarrow$

So, no such a exists.

So, $\{u\}$ is not linearly dependent.