Consider $\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \cdots \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \in F^{-1}$

clearly every $v \in F^n$ can be written in a unique way as a linear comb

of those $\begin{pmatrix}2\\-7\\8\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + (-7)\begin{pmatrix}0\\1\\0\end{pmatrix} + 8\begin{pmatrix}0\\0\\1\end{pmatrix}$

we will show that every vector space has such a "basis" - a collection of vecto

with which you can write every other vector in a unique way

lectrue
03.10.06.

Def: $v \in V$ is a linear comb of elements in $S \subset V$ if $\exists\ u_1, \dots, u_n\ S$

& $a_1, \dots, a_n \in F$ s.t. $v = \sum a_i u_i$

Example: In $P_3(\mathbb{R}) =$ (Polynomials of degree ≤ with coeff's in $\mathbb{R}$)

$v_1 = 2x^3 - 2x^2 + 12x - 6$ is a l.c. of

$u_1 = x^3 - 2x^2 - 5x - 3$ & $u_2 = 3x^3 - 5x^2 - 4x - 9$

but $V_2 = 3x^3 - 2x^2 + 7x + 8$ is not

Why?

$$2x^3 - 2x^2 + 12x - 6 \stackrel{?}{=} v_1 = a_1u_1 + a_2u_2$$
$$= a_1(x^3 - 2x^2 - 5x - 3) + a_2(3x^3 - 5x^2 - 4x - 9)$$

Equal coef for x^3: $2 = a_1 + 3a_2$

x^2: $-2 = -2a_1 - 5a_2$

x^1: $12 = -5a_1 - 4a_2$

x^0: $-6 = -3a_1 - 9a_2$

2 first + second

$2 = a_2,$

in first $2 = a_1 + 3 \cdot 2 \Rightarrow a_1 = -4$

$x^1: \ 12 = 20 - 4 \cdot 2$ ✓

now check for other equations if they hold. $x^0: \ -6 = -3 \cdot (-4) - 9 \cdot 2$ ✓

Check that V_2 isn't a l.c. of u_1 & u_2

Equal coeff's for
$$
\begin{aligned}
x^3 &: \quad 3 = a_1 + 3 \\
x^2 &: \quad -2 = -2a_1 - 5a_2 \\
x^1 &: \quad 7 = -5a_1 - 4a_2 \\
x^0 &: \quad 8 = -3a_1 - 9a_2
\end{aligned}
$$

2 first + second $\quad 4 = a_2 \quad \Rightarrow \quad a_2 = 4$

$3 = a_1 + 3 \cdot 4 \qquad a_1 = -9$

check for other equations if they hold. $7 \neq (-5)(-9) - 4 \cdot 4$

Example: show that $v_1 = \binom{2}{3}$ in $\mathbb{R}^2$ is a l.c. of $u_1 = \binom{1}{0}$, $u_2 = \binom{0}{1}$ & $u_3 = \binom{1}{1}$

Indeed $v_1 = 2u_1 + 3u_2 + 0u_3$

$\therefore$ coeff's are not unique

$= 0u_1 + 1u_2 + 2u_3$

Def:

we say that a subset $S \subset V$ "generates" or "spans" V if

$\text{span } S = \left\{ \begin{matrix} \text{all l.c. of} \\ \text{elements on } S \end{matrix} \right\} = V$

Examples: $V = M_{2\times 2}(\mathbb{R})$ $\quad M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$N_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $N_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $N_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $N_4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Claims: 1. $\{M_1, M_2, M_3, M_4\}$ generates V

2. $\{N_1, N_2, N_3, N_4\}$ generates V

3. $\{M_1, M_2, M_3\}$ does not generate V

4. $\{N_1, N_2, N_3\}$ does not generate V

Prf of 1. Given any $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ need to find $a_1 \dots a_4$ such that

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = B = a_1 M_1 + a_2 M_2 + a_3 M_3 + a_4 M_4 = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a_3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & a_4 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \Leftrightarrow \begin{aligned} b_{11} &= a_1 \\ b_{12} &= a_2 \\ b_{21} &= a_3 \\ b_{22} &= a_4 \end{aligned}$$

Proof of 2: Given any $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ need to find $a_1 \dots a_4$ such that

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = B = a_1 N_1 + a_2 N_2 + a_3 N_3 + a_4 N_4 = \begin{pmatrix} 0 & a_1 \\ a_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & 0 \\ a_2 & a_2 \end{pmatrix} + \begin{pmatrix} a_3 & a_3 \\ 0 & a_3 \end{pmatrix} + \begin{pmatrix} a_4 & a_4 \\ a_4 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_2+a_3+a_4 & a_1+a_3+a_4 \\ a_1+a_2+a_4 & a_1+a_2+a_3 \end{pmatrix} \Leftrightarrow \left.\begin{aligned} b_{11} &= a_2+a_3+a_4 \\ b_{12} &= a_1+a_3+a_4 \\ b_{21} &= a_1+a_2+a_4 \\ b_{22} &= a_1+a_2+a_3 \end{aligned}\right\} \text{sys of 4 eq'n with 4 unknowns}$$

trick

How to find M_1: $(N_1+N_2+N_3+N_4) - 3N_1 = 3M_1$ so $M_1 = -\frac{2}{3}N_1 + \frac{1}{3}N_2 + \frac{1}{3}N_3 + \frac{1}{3}N_4$

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} - 3\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow M_2 = \frac{1}{3}(N_1 + N_2 + N_3 + N_4 - 3N_2)$

$\vdots$

$B = b_{11}M_1 + b_{12}M_2 + b_{21}M_3 + b_{22}M_4$

Fact: $T \subset \operatorname{Span} S$

$\Rightarrow \operatorname{Span} T \subset \operatorname{Span} S$

M_i's $\quad N_i$

$\cap \quad \parallel$

lin. comb. of lin comb of elements in S $\subset$ lin. comb of elements in S

$= b_{11}\left(\frac{-2}{3}N_1 + \frac{1}{3}N_2 + \frac{1}{3}N_3 + \frac{1}{3}N_4\right) + b_{12}\left(\frac{1}{3}N_1 - \frac{2}{3}N_2 + \frac{1}{3}N_3 + \frac{1}{4}N_4\right) + \cdots =$ a l.c of $N_1 \cdots N_4$ ∎

Inde: in

$a_1M_1 + a_2M_2 + a_3M_3 = \begin{pmatrix} a_1 & a_2 \\ a_3 & 0 \end{pmatrix}$ lower right is always 0 so $\begin{pmatrix} 240 & 152 \\ e & \pi \end{pmatrix}$ not in span (since $\pi \neq 0$)

$a_1N_1 + a_2N_2 + a_3N_3 = \begin{pmatrix} a_2+a_3 & a_1+a_3 \\ a_1+a_2 & a_1+a_2+a_3 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 240 & 152 \\ e & \pi \end{pmatrix} = \begin{aligned} 240 &= a_2+a_3 \\ 152 &= a_1+a_3 \\ e &= a_1+a_2 \\ \pi &= a_1+a_2+a_3 \end{aligned} \Longrightarrow$ No solution

Motivation

$S \subset V$ is linearly dependent if it is "wasteful"; i.e, if $\exists v \in V$

s.t. $\exists\ a_1 \ldots a_n \in F$ & $u_1 \ldots u_n \in S$

& $\exists\ b_1 \ldots b_n \in F$ & $w_1 \ldots w_m \in S$

so that $\displaystyle\sum_{i=1}^{n} a_iu_i = v = \sum_{i=1}^{m} b_iw_i \Rightarrow \underbrace{\sum a_iu_i - \sum b_iw_i = 0}$

$$\boxed{\sum c_iz_i = 0}$$