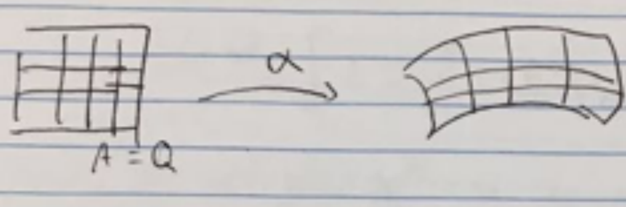
Jan. 16th.

Def: A parametrized K-manifold in $\mathbb{R}^n$ is a C^1 map $\alpha: A \to \mathbb{R}^n$ where $A \subset \mathbb{R}^k$ is some open set

$\alpha(A) = Y$ (the manifold) α (the parametrized)



pick a partition P of A

$$\mathrm{Vol}(Y) \sim \sum_{R \in P} \mathrm{Vol}(\alpha(R)) \sim \sum_{R \in P} V(D\alpha(c)(d_1 - c_1)e_1, D\alpha(c)(d_2-c_2)e_2, \ldots, D(\alpha)(c)(d_k - c_k)e_k)$$

$R = \Pi[c_i, d_i]$

$c = \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$

claim

$$V(x_1, \lambda x_i, \ldots x_k) = |\lambda| \cdot V(x_1 \ldots x_k)$$

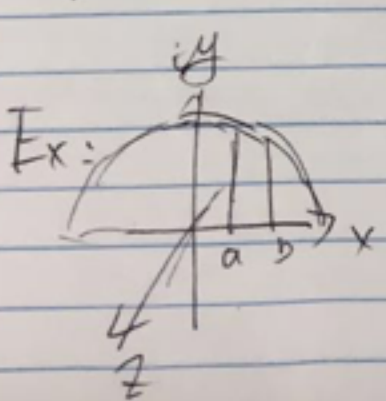
$$= \sum_{R \in P} \prod_{i=1}^{k} (d_i - c_i) \cdot \underbrace{V(D\alpha(c)e_1, \ldots D\alpha(c)e_k)}_{X = D\alpha(c)}$$

$$= \sum_{R \in P} \mathrm{Vol}(R) \cdot V(D\alpha(c))$$

$$\sim \int_Q V(D\alpha).$$

Def: $V(Y) = V(Y, \alpha) := \int_A V(D\alpha) = \int_A |\det((D\alpha)^T D\alpha)$

Ex:



$A = (a,b) \times (0, 2\pi)$.

$$\alpha \overset{(x,\theta)}{=} \begin{vmatrix} x \\ \sqrt{1-x^2}\cos\theta \\ \sqrt{1-x^2}\sin\theta \end{vmatrix}$$

$$\alpha(x) = \begin{bmatrix} 1 & 0 \\ \frac{-x\cos\theta}{\sqrt{1-x^2}} & -\sqrt{1-x^2}\sin\theta \\ \frac{-x\sin\theta}{\sqrt{1-x^2}} & \sqrt{1-x^2}\cos\theta \end{bmatrix} \qquad (D\alpha)^T D\alpha = \begin{pmatrix} \frac{1}{1-x^2} & 0 \\ 0 & 1-x^2 \end{pmatrix}$$

$$|\det (D\alpha)^T D\alpha|^{\frac{1}{2}} = 1$$

$$Vol(Y = \int_A 1 = 2\pi(b-a).$$

Thm: If $\alpha: A \to \mathbb{R}^n$ is a parametrized manifold

$g: B \to A$ a diffeomorphism, then $\beta = \alpha \circ g$

then $\alpha(A) = \beta(B)$ and

$$V(Y,\alpha) = V(Y,\beta)$$

By CoV.

$$\int_B D(\alpha\circ g) = \int_B (D\alpha\, Dg)^T (Dg \circ D\alpha)$$

By Chain Rule

$$= \int_B A.$$