

Jan 9<sup>th</sup> Agenda: TT2 much like TT1, details on Wed.

Today: Isometries, Gram-Schmidt, k-vols in  $\mathbb{R}^n$

Read along: Sec 20-21

Riddle 

A write 1-18

B picks one of the 3 dice

A picks another

Throw out remaining

Play dice war 1000 times.

Def:  $h: \mathbb{R}^n \rightarrow \mathbb{R}^n$  isometry  $\Leftrightarrow \forall x, y \quad d(hx, hy) = d(x, y)$

Thm:  $h$  is an isometry iff it is of the form  $hx = P + Ax$

where  $P \in \mathbb{R}^n$ ,  $A \in M_{n \times n}$  s.t.  $A^T A = I$

comments: Such  $h$  is "volume preserving"

comments 2: A linear transformation  $A$  s.t.  $A^T A = I$  is called "orthogonal".

This means  $A^T A = I \Leftrightarrow \langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

$\Rightarrow v_i \perp v_j$  if  $i \neq j$

&  $\|v_i\| = 1$  for any  $i \Rightarrow \{v_i\}$  forms an

orthonormal basis"

$\rightarrow A$  maps the standard basis to an orthonormal basis

" $A$  is a standard rotation"

comment 3: Rotation matrices / Orthogonal matrices

$$O(n) = \{A \in M_{n \times n}(\mathbb{R}) : A^T A = I\}$$

"form a group"

1.  $A, B \in O(n)$ ,  $AB \in O(n)$ ,  $A(BC) = (AB)C$ .

2.  $\exists I \in O(n)$  s.t.  $AI = IA = A \quad \forall A \in O(n)$

3.  $\forall A \in O(n)$ ,  $\exists B \in O(n)$  s.t.  $AB = BA = I$

pt 1 if  $A$  &  $B$  satisfy  $A^T A = I = B^T B$ ,  
is  $AB \in O(n)$ ?

$$(AB)^T AB = B^T A^T AB = B^T I B = B^T B = I. \checkmark$$

2. take  $I = I_{nn} \in O(n)$ ?

$$I^T I = I \cdot I = I^2 = I$$

3.  $A^T A = I \Leftrightarrow A^T = A^{-1}$ , is  $A^T \in O(n)$ ?

$(A^T)^T A^T = AA^T$ . For a square matrix, the left inverse

is also the right inverse.

$$\Rightarrow AA^T = I$$

Pf of Thm!  $\Leftarrow$  given  $h(x) = P + Ax$ ,  $A \in O(n)$ ,

$$\begin{aligned}d(h(x), h(y)) &= \|h(x) - h(y)\| = \|P + Ax - (P + Ay)\| \\&= \|Ax - Ay\| = \|A(x-y)\| = \sqrt{\langle A(x-y), A(x-y) \rangle} \\&= [(A(x-y))^T A(x-y)]^{1/2} = [(x-y)^T \cdot A^T A \cdot (x-y)]^{1/2} \\&= [(x-y)^T (x-y)]^{1/2} = \langle x-y, x-y \rangle^{1/2} = \|x-y\| = d(x, y)\end{aligned}$$

$\Rightarrow$  1. WLOG,  $h(0) = 0 \rightarrow P = 0$

indeed,  $h$  is an isometry, Pf  $h(x) := h(x) - h(0)$ .

so  $h(x) = Ax$  for  $A \in O(n)$ .

so  $h(x) = h(0) + Ax$

2.  $h$  "preserves norms"

$$d(h(x), 0) = \|h(x)\| = \|x\|, \quad d(h(x), h(0)) = d(x, 0)$$

3.  $h$  preserves inner products

$$\langle h(x), h(y) \rangle = \langle x, y \rangle$$

$$\begin{aligned}\|x-y\|^2 &= \langle x-y, x-y \rangle = \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle \\&= \|x\|^2 - 2\langle x, y \rangle + \|y\|^2\end{aligned}$$

$$\|h(x) - h(y)\|^2 = \|h(x)\|^2 - 2\langle h(x), h(y) \rangle + \|h(y)\|^2$$

$$0 = 2\langle x, y \rangle - 2\langle h(x), h(y) \rangle$$

$$\langle x, y \rangle = \langle h(x), h(y) \rangle$$

4. Set  $A = \begin{pmatrix} | & | & & | \\ h(e_1) & h(e_2) & \dots & h(e_n) \\ | & | & & | \end{pmatrix}$

Claim:  $A \in O(n)$

$$\begin{aligned}(A^T A)_{ij} &= \langle h(e_i), h(e_j) \rangle \\&= \langle e_i, e_j \rangle = \delta_{ij}\end{aligned}$$

$$\Rightarrow A^T A = I$$