2. S "linearly independent" $\Longleftrightarrow$

$\left(\sum \alpha_i u_i = 0,\ u_i \in S \text{ distinct} \Rightarrow \forall i, \alpha_i = 0 \right)$

Comments:

1. $\emptyset$ is lin. indep

2. $\{u\}$ is lin indep iff $u \neq 0$

$u \neq 0 \Rightarrow [\alpha u = 0 \Rightarrow \alpha = 0]$.

$u = 0 \Rightarrow 1 \cdot u = 0 \Rightarrow \{u\}$ is lin. dep.

3. Suppose $S_1 \subset S_2 \subset V$, Then

a. If S_1 is lin. dep, then so is S_2

b. If S_2 is lin dep, then so is S_1

4. If S is lin indep & $v \in V$, $v \notin S$, then $S \cup \{v\}$ is lin dep iff $v \in \operatorname{span}(S)$

Pf If $v \in \operatorname{Span} S$ then $\exists \alpha_i\ u_i$ s.t.

$v = \sum \alpha_i u_i$

$\Rightarrow 1 \cdot v - \sum (\alpha_i u_i) = 0.$

this is a linear comb of distinct vector

in $S \cup \{v\}$ & at least one scalar is nonzero

So $S \cup \{v\}$ is dep.

Suppose $S \cup \{v\}$ is lin dep $\Rightarrow$

$\exists$ a non-triv comb of elements of

$S \cup \{v\}$ equal to 0

$\Rightarrow \exists \alpha_i, u_i \in S$ (distinct) $\beta \in F$ s.t.

$\sum \alpha_i u_i + \beta v = 0.$

& not all $\{\alpha_i, \beta\}$ non 0.

If $\beta = 0$ then $\sum \alpha_i u_i = 0$ is a nontrivial comb of elements of S equal to 0, but S is indep

So $\beta \neq 0$, Then.

$\beta v = -\sum \alpha_i u_i$

$v = -\sum \beta^{-1} \alpha_i u_i$.

So $v \in \text{Span}(S')$

$\square$

Def A subset $\beta \subset V$ is called a "basis" if β is lin. indep and $\text{span}(\beta) = V$

Comment: often it is useful to talk about "ordered basis".

$\beta = \{u_1, \cdots, u_n\}$.

Thm A subset $\beta \subset V$ is a basis iff every $v \in V$ can be expressed in a unique way as a l.c of elements of β

PF: Suppose β is a basis, $\text{span}(\beta) = V$

every v can be expressed as a l.c.