$\mathcal{F}(S,\mathbb{R}) = \{ f : S \to \mathbb{R} \}$ $f(s) = 0$ $\forall s \in S$. $f \in \mathcal{F}(S, \mathbb{K})$.

row. e. form is unique where e. fors is not.

Lecture 16.10.06

$$2x - 7y = -3$$
$$-3x + 2y = -4$$

$$m \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right\}$$ system of lin. eq'ns

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

$\boxed{Ax = b}$ $\Leftrightarrow$ our system

$$x - y = 7$$
$$x + 3y = 2$$

$x \longmapsto \begin{pmatrix} x \\ y \end{pmatrix}$ $b \longmapsto \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

Lucky Case: A is square & invertible

our system

$$Ax = b \Leftrightarrow \underbrace{A^{-1}A}_{} x = A^{-1} \cdot b$$
$$I x = A^{-1} \cdot b$$
$$\Rightarrow \boxed{x = A^{-1} \cdot b}$$ lucky case

Theory:

$b=0$ "homogeneous system"

Claim: Consider $AX=0$ $A \in M_{n\times m}(F)$ also a lin. trans.

$A: F^n \longrightarrow F^m$

① $X=0$ is always a solution

② X is a solution iff $X \in$ null space (A)
$=$ kernel (A) : $Ker(A)$

$b \neq 0$ "non-homogeneous system"

claim: Consider $Ax=b$

① System has a solution iff $b \in$ range $(A) = Im\, A$

② If X_0 solves the system $(AX_0=b)$

X_1 solves system iff $X_1 = X_0 + X$ where x solves the homogeneous part of the matrix. $\Rightarrow$ $\boxed{AX=0}$

Proof: We know $Ax_0=b$

$Ax_1=b \Leftrightarrow A(x_0+X)=b \Leftrightarrow Ax_0+Ax=b$

$\Leftrightarrow b+Ax=b \Leftrightarrow Ax=0$

We knew

$x_1 = x_0 + x$ (homogeneous part of the matrix)

$X = X - X_0$

row operations = multiplying elem. matrices on the left

With row reduction:

$$A \rightsquigarrow A' = \underbrace{E}_{\text{a product of invert. matrices, hence invertible.}} \cdot A \quad \text{"in r.r.e.f"}$$

$$Ax = b \iff \underbrace{EA}_{A'}x = \underbrace{Eb}_{b'} \iff A'x = b'$$

$$\Updownarrow$$

$$T(Ax) = T(b)$$

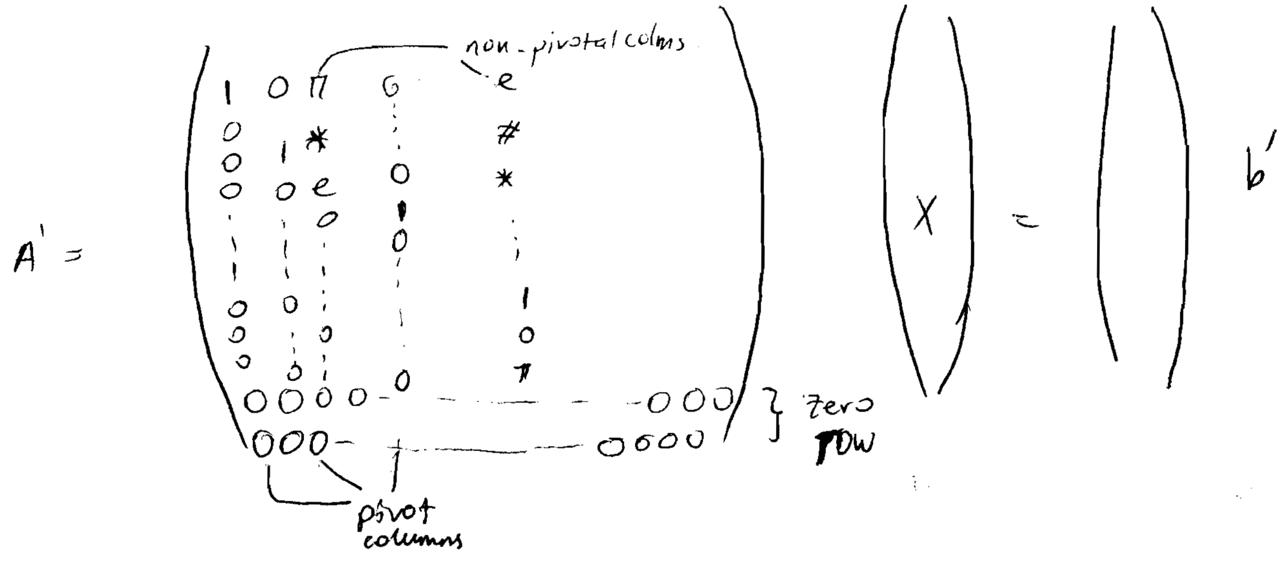
Mechanical Aside: To find A' & b' at the same time,

form $\tilde{A} = (A/b)$ row-reduce it

$$E\tilde{A} = (\underbrace{EA}_{A'}/\underbrace{Eb}_{b'})$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad \tilde{A} = \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & -1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & -2 & -5 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5/2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 9/2 \\ 0 & 1 & 5/2 \end{array}\right)$$



Thm: ① If b' has a non-zero entry in one of the zero rows of A', no solution.

② Otherwise all solutions are obtained by setting the entries of x corresponding to the non-pivotal col's in an arbitrary way, and then the rest is uniquely determined by solving single lin. eqn's.

Tutorial:

Defn: $Ax=b$, $b, x \in V^n$, $A \in M_{n,n}(V)$ if $b=0$, then this system is called homogeneous.

Thm: If the system $T_A x = 0$ then is homog. then the dimention k of the space of sol's. of that system

$k = n - \text{rank}\, T_A = n - \text{rank}\, A$

dimension of the kernel of T_A (nullity of T_A)

- max # of lin. indep. columns in $A = \text{rank}(T_A) =$ max # of lin. indep. rows in A.

1) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$T(a,b,c) = (a+b, 2a-c)$

Determine $T^{-1}(1,11)$ i.e. the set of preimages of the point $(1,11)$