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Part 2

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10:07 AM

① $xy \in G$, let $|xy| = n$ and $|yx| = k$

Claim: If $x, y \in G$ then $|xy| = |yx|$

$$e = (xy)^n = x(yx)^{n-1}y \quad * \text{ associativity}$$

$$\Rightarrow x^{-1} = (yx)^{n-1}y$$

$$\Rightarrow x^{-1}x = (yx)^{n-1}yx$$

$$\Rightarrow e = (yx)^n$$

Thus $|yx|$ divides $|xy|$. Conversely,

$$e = (yx)^k = y(xy)^{k-1}x$$

$$\Rightarrow y^{-1} = (xy)^{k-1}x$$

$$\Rightarrow y^{-1}y = (xy)^{k-1}xy$$

$$\Rightarrow e = (xy)^k$$

So $|xy|$ divides $|yx|$! Since they both divide each other,

$$|xy| = |yx|.$$

② <u>Claim:</u> $\phi(g) : G \to G$ is a homomorphism
$g \mapsto g^2$

iff G is abelian

<u>Proof</u>

($\Rightarrow$) Let $g_1, g_2 \in G$.

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$$

$$(g_1 g_2)^2 = g_1^2 g_2^2$$

$$g_1 g_2 g_1 g_2 = g_1 g_1 g_2 g_2$$

$$\Rightarrow g_2 g_1 = g_1 g_2$$

($\Leftarrow$) G is abelian $\Rightarrow g_1 g_2 = g_2 g_1$

$$\phi(g_1 g_2) = (g_1 g_2)^2$$
$$= g_1 g_2 g_1 g_2$$
$$= g_1 g_1 g_2 g_2$$
$$= g_1^2 g_2^2$$
$$= \phi(g_1)\phi(g_2)$$

Thus, ϕ is a homomorphism.

③ let $[a,b] = aba^{-1}b^{-1}$

And the commutator subgroup be

$$G' = \{ \langle aba^{-1}b^{-1} \rangle \mid a,b \in G \}$$

Claim: G' is normal in G.

let $a, b, \tilde{a}, \tilde{b}, g \in G$ with

$$\tilde{a} = a^g$$
$$\tilde{b} = b^g$$

Lemma: $[a,b]^g = [a^g, b^g]$

pf

For any $g' = [a,b] \in G'$

$$(g')^g = g^{-1} aba^{-1}b^{-1} g$$
$$= g^{-1}agg^{-1}bgg^{-1}a^{-1}gg^{-1}b^{-1}g$$
$$= \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} \in G'$$

Since $[a,b]^g = [a^g, b^g]$

Lemma: Finite products of elements of G' conjugate to G'.

$$\left[\prod_{i=1}^{n} [a_i, b_i]\right]^g = g^{-1}\left[\prod_{i=1}^{n}[a,b]\right] g$$
$$= \prod_{i=1}^{n}\left[g^{-1}[a,b]g\right] \quad * \; gg^{-1} = e$$
$$= \prod_{i=1}^{n}[a^g, b^g]$$
$$\in G'$$

$\in G'$

Thus, $G' \triangleleft G$.

Claim $\frac{G}{G'}$ is abelian.

Pf: Let $[g_1], [g_2] \in \frac{G}{G'}$

$$[g_1]\cdot[g_2] = [g_1(g_1^{-1}g_2g_1g_2^{-1})]\cdot[g_2] \quad * \; g_1^{-1}g_2g_1g_2^{-1} \in G'$$

$$= [g_1g_1^{-1}g_2g_1g_2^{-1}g_2]$$

$$= [g_2g_1]$$

$$= [g_2]\cdot[g_1]$$

Therefore G/G' is an abelian group.

Claim: Let $\pi: G \to G/G'$, $g \mapsto [g]$. For any homomorphism $\varphi: G \to A$ with A an abelian group, $\exists$ a homomorphism $\bar{\varphi}: \frac{G}{G'} \to A$ such that $\varphi = \bar{\varphi} \circ \pi$.

Pf: If $g \in G$, then $[g] \in \frac{G}{G'}$ with

$$\bar{\varphi}([g]) = \varphi(g)$$

First, $G' \subset \ker\varphi$ because for any generator $g' = aba^{-1}b^{-1} \in G'$ we have:

$$\varphi(g') = \varphi(aba^{-1}b^{-1})$$

$$= \varphi(a)\varphi(b)\varphi(a^{-1})\varphi(b^{-1}) \quad * \text{ Homomorphism property}$$

$$= \varphi(a)\varphi(a^{-1})\varphi(b)\varphi(b^{-1}) \quad * \; \varphi \text{ maps to an Abelian group}$$

$$= e$$

I will show that $\bar{\varphi}$ is well defined. Let $g, h \in G$. Suppose $[g] = [h]$, then $gG' = hG' \Rightarrow g = hx$ for some $x \in G'$.

$$\bar{\varphi}([g]) = \varphi(g)$$

$$= \varphi(hx) \quad * \text{ same equivalence class}$$

$$= \varphi(h)\varphi(x) \quad * \text{ homomorphism}$$

$= \varphi(h)\,\varphi(x)$ * homomorphism

$= \varphi(h)\, e$ $x \in G'$ and $G' \subset \ker\varphi$

$= \overline{\varphi}([h])$

So $\overline{\varphi} : \frac{G}{G'} \to A$ is a well defined map!

I will show that $\overline{\varphi}$ is a homomorphism.
Let $g_1, g_2 \in G$, then

$\overline{\varphi}([g_1]\cdot[g_2]) = \overline{\varphi}([g_1 g_2])$ * Group operation on cosets

$= \varphi(g_1 g_2)$ * Property of $\overline{\varphi}$

$= \varphi(g_1)\varphi(g_2)$ * φ is a homomorphism

$= \overline{\varphi}([g_1])\,\overline{\varphi}([g_2])$

Thus, any $\varphi : G \to A$ with A abelian must factor through $\frac{G}{G'}$!

4 $\quad \mathrm{Aut}\, G = \{ \phi : G \to G,\ \phi^{-1} \text{ exists} \}$

Claim: $\mathrm{Inner}\, G \trianglelefteq G$.

First, prove $\mathrm{Inner}\, G \leqslant \mathrm{Aut}\, G$

Let $\phi^g, \phi^{h^{-1}}$ be conjugation by g and h^{-1}

$$\phi^g \circ \phi^{h^{-1}}(x) = \phi^g(h x h^{-1})$$
$$= g^{-1} h x h^{-1} g$$
$$= (h^{-1}g)^{-1} x\, h^{-1} g$$
$$= \phi^{h^{-1}g}(x)$$
$$\in \mathrm{Inner}\, G$$

So in fact, $\mathrm{Inner}\, G \leqslant \mathrm{Aut}\, G$

Now show that $\mathrm{Inner}\, G \trianglelefteq \mathrm{Aut}\, G$.

Let $\phi \in \mathrm{Aut}\, G$, $\phi^g \in \mathrm{Inner}\, G$.

$$\phi^{-1} \circ \phi^g \circ \phi(x) = \phi^{-1} \circ \phi^g(\tilde{x})$$

$$= \phi^{-1}(g^{-1}\tilde{x}g)$$

$$= \phi^{-1}(g^{-1})\,\phi^{-1}(\tilde{x})\,\phi^{-1}(g)$$

$$= \tilde{g}^{-1}\,x\,\tilde{g}$$

$$= \phi_{\tilde{g}}(x)$$

$$\in \text{Inner } G$$

So $\text{inner } G \trianglelefteq \text{Aut } G$.