

Homework 7. TUT: R 4-5 p.m.

Section 1).

2. prove by contradiction.

Assume there exist a open set  $E$  (non-trivial)  $\subset \mathbb{R}^n$ , has measure zero in  $\mathbb{R}^n$ .

Since  $E$  is open & let  $x \in E$ .

so.  $\exists \epsilon > 0$ . s.t.  $V(x, \epsilon) \subset E$ .

By the THM 11.1(c): if  $B \subset A$  &  $A$  has measure zero in  $\mathbb{R}^n$ , then so does  $B$ .

So.  $V(x, \epsilon)$  is measure zero in  $\mathbb{R}^n$ .

(c) for every  $\epsilon > 0$ , there is a covering  $Q_1, Q_2, \dots$  of  $V(x, \epsilon)$  by countably many rectangles s.t.  $\sum_{i=1}^{\infty} V(Q_i) < \epsilon$ .

Since for every  $y, z \in V(x, \epsilon)$ .  $d(y, z) \leq d(y, x) + d(x, z) < \epsilon + \epsilon = 2\epsilon > 0$ .

so  $\sup(d(y, z)) = 2\epsilon > 0$ .

Hence  $\exists \delta > 0$ . s.t.  $V(V(x, \epsilon)) = \delta > 0$ .

we can take  $G = \frac{\delta}{2}$ . so.  $G < \delta = V(V(x, \epsilon)) \leq \sum_{i=1}^{\infty} V(Q_i)$   $\Rightarrow$

contradiction to.  $V(x, \epsilon)$  is measure zero in  $\mathbb{R}^n$ .

4. Let.  $Q = [0, 1] \cap \mathbb{Q}$  = rationals in  $[0, 1]$ .

so.  $Q^c = [0, 1] \setminus Q$  = irrationals in  $[0, 1]$ .

$\Rightarrow Q \cup Q^c = [0, 1]$

claim:  $Q$  has measure zero in  $\mathbb{R}$ .

pf. given  $\epsilon > 0$ , let  $q_i$  be a listing of the element of  $Q$ .  $Q = \{q_i\}$

take.  $R_i = [q_i - \frac{\epsilon}{2^{i+1}}, q_i + \frac{\epsilon}{2^{i+1}}]$ .  $V(R_i) > V(Q)$ .

$$\sum_{i=1}^{\infty} V(R_i) = \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon < \epsilon.$$

So  $Q$  has measure zero in  $\mathbb{R}$ .

by claim.  $Q$  is rationals in  $[0, 1]$  is measure zero in  $\mathbb{R}$ .

claim 2:  $[0, 1]$  not have measure zero in  $\mathbb{R}$ .

pf. Assume  $A_1, A_2, \dots$  are countably many rectangles cover  $[0, 1]$ .

Since  $[0, 1]$  is compact.  $\exists k > 0$ . s.t.  $A_1, \dots, A_k$  covers  $[0, 1]$ .

$$\text{so } \sum_{i=1}^k V(A_i) > V([0, 1]) = 1.$$

$$\text{let } G = \frac{1}{k}. \quad \sum_{i=1}^k V(A_i) > \sum_{i=1}^k V(A_i) > 1 > \epsilon.$$

so.  $[0, 1]$  not have measure zero in  $\mathbb{R}$ .

Assume.  $Q_i^c$  has measure zero in  $\mathbb{R}$ .

Since  $Q$  has measure zero in  $\mathbb{R}$  by claim 1, they  $[0, 1] = Q \cup Q_i^c$  has measure 0 in  $\mathbb{R}$

by THM 11.1(b). This contradicts to the claim 2. s.t.  $[0, 1]$  does not have measure 0 in  $\mathbb{R}$ .

Therefore.  $Q_i^c$  does not have measure in  $\mathbb{R}$ .

6. ① show  $f$  is cont.

Given  $\epsilon > 0$ , there  $\exists$  a integer,  $n$ , st. if  $x, y \in [a, b]$ , &  $|x-y| < \frac{b-a}{n}$ .  
then  $|f(x) - f(y)| < \epsilon$ .  $\Rightarrow f$  is cont.

② Since  $f$  is cont. on a compact set  $[a, b]$ , so  $f$  is uniformly cont.  
then  $f$  is integrable on  $[a, b]$ .

By the definition of integrable,  $\forall \epsilon > 0$ ,  $\exists$  a partition  $P = \{I_1, I_2, \dots, I_n\}$  on  $[a, b]$ .  
st.  $U(f, P) - L(f, P) < \epsilon$ .

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$$U(f, P) = \sum_{i=1}^n M(I_i) \cdot m_i(f), \quad L(f, P) = \sum_{i=1}^n m(I_i) \cdot M_i(f) \quad [m(I_i) \text{ means the length of } I_i]$$
$$U(f, P) - L(f, P) = \sum_{i=1}^n M(I_i)(M_i(f) - m_i(f)) < \epsilon.$$

Since  $f$  is cont. on  $I_i$ ,  $1 \leq i \leq n$ .  $\exists x_i, y_i \in I_i$ , st.  $f(x_i) = M_i(f)$ ,  $f(y_i) = m_i(f)$   
since  $G_f = \{(x, y) | y = f(x)\}$ , is the graph of  $f$ .

so the portion  $G_i$  of the graph  $f$ , with  $x \in I_i$ , is contained in the rectangle.  $\square$

$Q_i = I_i \times [m_i(f), M_i(f)] = l(I_i) \cdot (M_i(f) - m_i(f))$   $1 \leq i \leq n$ .

Hence,  $G_f$  is contained in the  $\bigcup_{i=1}^n Q_i$  since  $x \in [a, b] = \bigcup_{i=1}^n I_i$

$$\bigcup_{i=1}^n Q_i = \sum_{i=1}^n l(I_i)(M_i(f) - m_i(f)) < \epsilon. \quad (\text{we get it from above by } f \text{ is integrable})$$

Since this is true for all  $\epsilon > 0$ , st.  $\exists$  countable many rectangles  $Q_i$  covers  $G_f$ .  
With  $\sum_{i=1}^n V(Q_i) < \epsilon$ . So  $G_f$  has measure zero in  $\mathbb{R}^2$ .

8. ① we want to show  $\int f$  exists. ②  $f$  is integrable

Let  $Q = I_1 \times I_2 \times \dots \times I_n$ . Since  $f$  is bounded  $\exists M > 0$ . st.  $|f(x)| \leq M$ . ③  $M$  fix  $\in M$ .

Let  $\epsilon > 0$ . Since  $B$  of measure zero. So  $\exists$  closed rectangles  $Q_1, Q_2, \dots$  st.

$$B \subset \bigcup_{k=1}^{\infty} \text{int}(Q_k) \text{ & } \sum_{k=1}^{\infty} V(Q_k) < \frac{\epsilon}{2M}.$$

Since  $B$  is closed & bounded, so  $B$  is compact. ④  $\exists N$ . st.  $B \subset \bigcup_{k=1}^N \text{int}(Q_k)$ ,  $\sum_{k=1}^N V(Q_k) < \frac{\epsilon}{2M}$

Let  $P_i$  be the partition of  $I_i$  by taking end points of  $I_i$  & end points of  $i$ th component of  $Q_k$  for  $1 \leq k \leq N$ .

Let  $P = \{P_1, P_2, \dots, P_n\}$  be the partition of  $Q$ .

So every  $Q_k$  is a union of some subrectangles in  $P_i$ .

Let  $R_k$   $1 \leq k \leq N$  be disjoint collections of subrectangles  $R$  s.t.  $B \subset \bigcup R_k$ .

st.  $R \subset Q_k$ ,  $\sum_{R \in R_k} V(R) \leq V(Q_k)$ .

If  $R$  is the subrectangle determined by  $P$  st.  $R \notin Q_k$ , then  $\inf_R f = \sup_R f = 0$ .

Since  $f$  is vanish except on  $B$ .

$$\text{so } L(f, P) = \sum_{R \in R_k} \sum_{R \in P} V(R) (\inf_R f) \geq -M \sum_{R \in R_k} \sum_{R \in P} V(R) \geq -M \sum_{R \in R_k} V(Q_k) > -M \cdot \frac{\epsilon}{2M} = -\frac{\epsilon}{2}.$$

$$U(f, P) = \sum_{R \in R_k} \sum_{R \in P} V(R) (\sup_R f) \leq M \sum_{R \in R_k} \sum_{R \in P} V(R) \leq M \sum_{R \in R_k} V(Q_k) < M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2}.$$

Hence,  $U(f, P) - L(f, P) < \frac{\epsilon}{2} - (-\frac{\epsilon}{2}) = \epsilon$ .  $\Rightarrow f$  is integrable. ⑤  $\int f$  exists.

Hilary

By the THM 11.3.(a):  $f: Q \rightarrow \mathbb{R}$ .  $Q$  be a rectangle in  $\mathbb{R}^n$ , assume  $f$  is integrable over  $Q$ . If  $f$  vanishes except on a set of measure zero. then  $\int_Q f = 0$ .  
 Therefore, by the THM.  $\int_Q f = 0$ .

9. a). Since  $f$  is integrable, then  $\int_Q f = \inf_P \{L(f, P)\}$ : where  $P$  is a partition of  $Q$ .  
 since  $f(x) \geq 0$  for all  $x$ . then  $m_R(f) = \inf\{f(x) | x \in R\} \geq 0$ .  
 then  $L(f, P) = \sum_R v(R) \cdot m_R(f) \geq \sum_R v(R) \cdot 0 \geq 0$ .  
 so.  $\inf_P \{L(f, P)\} \geq 0 \Rightarrow \int_Q f \geq 0$ .

b). Since  $f$  is integrable, the set of points at which  $f$  is discontin. measures 0.  
 Let  $x_0 \in Q$ ,  $x_0$  is the point where  $f$  is cont. & let  $y = f(x_0) > 0$   
 Since  $f$  is cont. at  $x_0$ , so  $\exists_{x_0 \in Q} \forall \epsilon > 0$  st. for all  $x \in Q'$ ,  $|f(x) - y| < \frac{\epsilon}{2}$   
 Let  $P$  be any partition of  $Q$  so  $Q'$  is one of the subrectangles determined by  $P$ .  
 then  $L(f, P) = \sum_{R \in P} v(R) \cdot m_R(f) \geq v(Q') \cdot m_{Q'}(f) > v(Q') \frac{y}{2} > 0$ .  
 Since  $f$  is integrable  $\int_Q f = \sup_P L(f, P) \geq L(f, P) > 0$ .