elegant pf of $(AB)C = A(BC)$

$$U \xrightarrow[C]{T} V \xrightarrow[B]{S} W \xrightarrow[A]{P} Z$$

$$P\circ(S\circ T) = (P\circ S)\circ T$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A\cdot B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B\cdot A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

<u>Proposition</u>

Given

$$V' \xrightarrow{Q} V \xrightarrow{T} W \xrightarrow{P} W'$$

V.S. & l.t. as above, with $Q$ & $P$ invertible

(enough that $Q$ is surjective (onto) & $P$ is injective (onetoone))

then $\operatorname{rank}(T) = \operatorname{rank}(P\circ T\circ Q)$

<u>PF</u>

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "P"] & \supseteq R(T) \arrow[d] \\ V' \arrow[u, "Q"] \arrow[r, "T' = PTQ"'] & W' & \supseteq R(T') \end{tikzcd}$$

"commutative diagram"

with basis $w_i \in W$ & choose $v_i \in V$ s.t. $w_i = T(v_i)$

set $w_i' = P(w_i)$

<u>Claim</u> $(w_i')$ is a basis of $R(T')$ So $\operatorname{rank}(T) = \operatorname{rank}(T')$

Need to show

1. $w_i' \in R(T')$

$w_i' = P(w_i) = PT(v_i) = PTQQ^{-1}(v_i)$

$= T'(Q^{-1}(v_i)) \in R(T')$

2. $w_i'$ span $R(T')$

Assume $w' \in R(T')$, namely $w' = T'(v')$ for some $v' \in V'$. Let $w = TQ(v') \in W$, $w = T(Q(v')) \in R(T)$

So $w = \sum \alpha_i w_i$ for some $\alpha_i$, now

$w' = T(v') = PTQ(v') = P(w) = \sum \alpha_i P(w_i)$
$= \sum \alpha_i w_i'$

3. $w_i'$ are lin. indep.

<u>Def.</u> If $A \in M_{m \times n}$, set $\text{rank}(A) := \text{rank}(T_A)$ where $T_A : F^n \to F^m$ as before

## Comment 1:

Given $T: V \to W$, basis $\beta$ (on $V$), basis $\gamma$ (on $W$)

$$\operatorname{rank}(T) = \operatorname{rank}[T]_\beta^\gamma$$

Pf. We have the following commutative diagram

$$\begin{tikzcd}
v_1 \arrow[d] & V \arrow[r, "T"] \arrow[d, shift left, "{[\ ]_\beta}"] & W \arrow[d, "P"', "{[\ ]_\gamma}"] \\
e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & F^n \arrow[r, "T_A"', "A = {[T]_\beta^\gamma}"] \arrow[u, shift left, "Q"] & F^m
\end{tikzcd}$$

$(e_1 \cdots e_n)$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$T_A(e_1) = A\begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}$$

$$\| $$

$$PTQ(e_1) = PT(Q(e_1))$$

$$= PT(v_1) = [Tv_1]_\gamma$$

$$= \text{the } 1^{st} \text{ Col of } [T]_\beta^\gamma.$$

Now, $\operatorname{rank}([T]_\beta^\gamma) = \operatorname{rank}(A) = \operatorname{rank}(T_A) = \operatorname{rank}(T)$. $\square$