

elegant pf of  $(AB)/C = A(BC)$

$$U \xrightarrow{\begin{matrix} T \\ C \end{matrix}} V \xrightarrow{\begin{matrix} S \\ B \end{matrix}} W \xrightarrow{\begin{matrix} P \\ A \end{matrix}} Z$$

$$P \circ (S \circ T) = (P \circ S) \circ T$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B \cdot A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

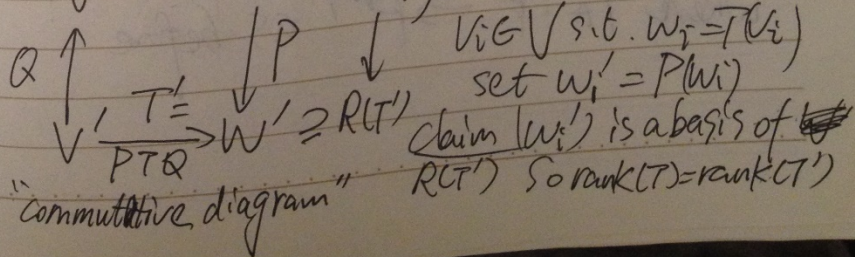
Proposition

Given  $Q$   
 $V' \xrightarrow{Q} V \xrightarrow{T} W \xrightarrow{P} W'$

V.S. & l.t. as above, with  $Q$  &  $P$  invertible  
(enough that  $Q$  is surjective (onto)  
&  $P$  is injective (one-to-one))

then  $\text{rank}(T) = \text{rank}(P \circ T \circ Q)$

PF  $V \xrightarrow{T} W \cong \text{R}(T)$  with basis  $w_i$  & choose



Need to show

1.  $w_i' \in R(T')$

$$w_i' = P(w_i) = PT(v_i) = PTQ(Q^{-1}(v_i)) \\ = T'(Q^{-1}(v_i)) \in R(T')$$

2.  $w_i'$  span  $R(T')$

Assume  $w' \in R(T')$ , namely  $w' = T'(v')$  for some  $v' \in V'$ . Let  $w = TQ(v') \in W$ ,  $w = T(Q(v')) \in R(T)$

so  $w = \sum d_i w_i$  for some  $d_i$ , AOW

$$w' = T(v') = PTQ(v') = P(w) = \sum d_i P(w_i) \\ = \sum d_i w_i'$$

3.  $w_i'$  are lin. indep.

Def. If  $A \in M_{m \times n}$ , set  $\text{rank}(A) := \text{rank}(T_A)$

where  $T_A: F^n \rightarrow F^m$  as before

Comment 1:

Given  $T: V \rightarrow W$   
basis  $\beta$       basis  $\gamma$

$$\text{rank}(T) = \text{rank} [T]_{\beta}^{\gamma}$$

PR. We have the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow Q \uparrow [I]_{\beta} & & \downarrow P \uparrow [I]_{\gamma} \\ \mathbb{F}^n & \xrightarrow{TA} & \mathbb{F}^m \end{array}$$

$T(e_i) = A \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix}$   
"  
 $P T Q(e_i) = P T(Q(e_i))$   
 $P T(v_i) = [T v_i]_{\gamma}$   
 $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = \text{the 1st col of } [T]_{\beta}^{\gamma}$

Now  $\text{rank} [T]_{\beta}^{\gamma} = \text{rank}(A) = \text{rank}(TA) = \text{rank}(T)$   $\square$