

Homework 8. TUT R 4-5 p.m.

Section 12.

3. a) Since f is bounded & integrable over \mathbb{Q} , then by the Fubini's THEOREM

$$\int f = \int_{x \in A} \int_{y \in B} f(x, y) = \int_{x \in A} \int_{y \in B} f(x, y), \text{ & } \int_{y \in B} f(x, y) \text{ & } \int_{x \in A} f(x, y) \text{ are integrable.}$$

(Let. $\int_{y \in B} f(x, y) = l(x)$. & $\int_{x \in A} f(x, y) = u(x)$. then $l(x) \leq g(x) \leq u(x)$, l, g, u are bounded)

claim: $\int_{x \in A} l(x) \leq \int_{x \in A} g(x)$.

Proof: Let P_1 be the partition on A of $l(x)$

$$\int_{x \in A} l(x) = \sup_{P_1} L(l, P_1) = \sup_{P_1} \left(\sum_{R \in P_1} V(R) m_R(l) \right).$$

$$\exists x_0 \in R, \text{ s.t. } l(x_0) \leq l(x) \text{ for all } x \in R \quad \Rightarrow \quad l(x_0) = m_R(l)$$

Since $l(x) \leq g(x)$ for all $x \in A$. then $l(x_0) \leq g(x)$ for all $x \in A$.

$$\text{so. } l(x_0) \leq m_R(g) \text{ for any rectangle } R \in A. \Rightarrow m_R(l) \leq m_R(g)$$

$$\Rightarrow \sup_{P_1} \left(\sum_{R \in P_1} V(R) m_R(l) \right) \leq \sup_{P_1} \left(\sum_{R \in P_1} V(R) m_R(g) \right).$$

Since $\int_{x \in A} g(x) = \sup_{P_2} L(g, P_2) = \sup_{P_2} \left(\sum_{R \in P_2} V(R) m_R(g) \right)$, P_2 is a partition of g .

① $\int_{x \in A} g(x)$ is the sup of $L(g, P_2)$, ② $\int_{x \in A} g(x) \geq L(g, P_2)$ for all Partition of P_2 on A .

$$\text{So. } \int_{x \in A} g(x) \geq \sup_{P_2} \left(\sum_{R \in P_2} V(R) m_R(g) \right). \text{ } P_2 \text{ is a kind of partition of } A.$$

$$\Rightarrow \int_{x \in A} g(x) \geq \int_{x \in A} l(x). \text{ as required.}$$

claim: $\int_{x \in A} g(x) \leq \int_{x \in A} u(x)$

Proof: Similarly, let P_2 be the partition on A of $u(x)$

$$\int_{x \in A} u(x) = \inf_{P_2} U(u, P_2) = \inf_{P_2} \left(\sum_{R \in P_2} V(R) M_R(u) \right).$$

$$\exists x_0 \in R, \text{ s.t. } g(x_0) \geq u(x) \text{ for all } x \in R. \quad M_R(g) = g(x_0).$$

Since $g(x) \leq u(x)$ for all $x \in A$. then $g(x_0) \leq u(x_0)$.

$$\exists x_0 \in R. \text{ s.t. } u(x_0) \geq u(x) \text{ for all } x \in R. \quad M_R(u) = u(x_0),$$

$$\Rightarrow M_R(u) \geq u(x_0) \geq g(x_0) \Rightarrow M_R(u) \geq M_R(g)$$

$$\inf_{P_2} \left(\sum_{R \in P_2} V(R) M_R(u) \right) \leq \sum_{R \in P_2} V(R) M_R(g) \text{ for any Partition of } A.$$

Let P_3 be the partition that has the $\inf U(u, P_3) = \int_{x \in A} u(x)$.

$$\text{then. } U(u, P_3) = \sum_{R \in P_3} V(R) M_R(u) \geq \sum_{R \in P_3} V(R) M_R(g) \geq \inf_{P_2} \sum_{R \in P_2} V(R) M_R(g)$$

Since P_3 is a partition on A .

$$\text{So. } \int_{x \in A} u(x) \geq \int_{x \in A} g(x). \text{ as required.}$$

Hence. $\int_{x \in A} l(x) \leq \int_{x \in A} g(x) \leq \int_{x \in A} u(x)$

Since $l(x)$ & $u(x)$ are integrable then. $\int_{x \in A} l(x) = \int_{x \in A} (l(x))$ & $\int_{x \in A} u(x) = \int_{x \in A} (u(x))$

& since. $\int f = \int_{x \in A} l(x) = \int_{x \in A} u(x)$, then. $\int f \leq \int_{x \in A} g(x) \leq \int_{x \in A} g(x) = \int f$

$$\text{So. } \int_{x \in A} g(x) = \int_{x \in A} g(x) = \int_A g = \int f$$

Therefore. g is integrable over A & $\int f = \int g$.



Hilary

$$b) f(x,y) = \begin{cases} 1 & x = \frac{1}{2} \text{ & } y \in \mathbb{Q} \\ 0 & \text{otherwise.} \end{cases} \quad (\text{let } Q = [0,1]^2)$$

$$f: Q \rightarrow \mathbb{R}$$

for any partition P , $m_P(f) = 0$ for R.E.P. so, $L(f, P) = 0 \Rightarrow \sup_P L(f, P) = 0$

$M_R(f) = 1$ only when $R = [\frac{1}{2}, S, \frac{1}{2} + S] \times [0, 1]$ we can make S small enough to make $V(R) > 0$
therefore $V(R) \cdot M_R(f) \geq 0$. then $\inf_P U(f, P) = 0$

Hence $\sup_P L(f, P) = \inf_P U(f, P) = 0 \Rightarrow \int_Q f \text{ exists}$

I consider $\int_{\mathbb{R}^2} f(x, y) d(x, y)$ (NOT exist)

(let x fixed & consider $\int_{\mathbb{R}} f(x, y) dy$)

$$\textcircled{1} \text{ if } x = \frac{1}{2} \quad f(x, y) = \begin{cases} 1 & y \in \mathbb{Q} \\ 0 & y \notin \mathbb{Q} \end{cases}$$

for any partition P on $[0, 1]$, we can find $m_P(f) = 0$ & $M_P(f) = 1$

so, $L(f, P) = 0$ but $U(f, P) = 1$ for any partition P .

then $\int_{\mathbb{R}} f(x, y) dy$ does not exist.

$\Rightarrow \int_{\mathbb{R}^2} f(x, y) d(x, y)$ does not exist.

II. Consider $\int_{\mathbb{R}^2} f(x, y) d(x, y)$ (Exist).

(let y fixed & consider $\int_{\mathbb{R}} f(x, y) dx$)

$$\textcircled{2} \text{ if } y \in \mathbb{Q}, \quad f(x, y) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \end{cases}$$

5/5

so, for any partition P on $[0, 1]$, we can find $m_P(f) = 0$, $L(f, P) = 0$.

only for the interval $[\frac{1}{2} - S, \frac{1}{2} + S]$, $M_P(f) = 1$ otherwise, $M_P(f) = 0$.

we can make $0 < S < \frac{\epsilon}{2}$ st. $\frac{1}{2} + S - (\frac{1}{2} - S) < \epsilon$. $U(f, P) = 1 \cdot 2S = 2S$.

Then, $U(f, P) - L(f, P) = 2S - 0 = 2S < \epsilon$.

by Riemann Cond.Thm, $\int_{\mathbb{R}} f(x, y) dx = 0$.

$\textcircled{2} \text{ if } y \notin \mathbb{Q}, \quad f(x, y) = 0$. then $\int_{\mathbb{R}} f(x, y) dx = 0$.

Hence $\int_{\mathbb{R}^2} f(x, y) d(x, y) = \int_{\mathbb{R}^2} 0 = 0$.

Therefore, since 0 is a constant. then $\int_{\mathbb{R}^2} 0 = 0 \Rightarrow \int_{\mathbb{R}^2} f(x, y) d(x, y) = 0$ exist.

c)

See Last Page for Answer

Problem A.

$$f(x,y,z) = \begin{cases} 1 & x < y < z \\ 0 & \text{otherwise} \end{cases} \quad x \in [0,1], \quad y \in [0,1], \quad z \in [0,1]$$

$$\int_0^1 \left[\int_0^z \left[\int_0^y 1 \, dx \right] dy \right] dz = \int_0^1 \left[\int_0^z \left(\int_0^y 1 \, dx \right) dy \right] dz = \int_0^1 \left(\int_0^z y \, dy \right) dz$$

$$= \int_0^1 \left(\frac{1}{2} y^2 \Big|_0^z \right) dz = \int_0^1 \frac{1}{2} z^2 dz - \int_0^1 \frac{1}{2} 0^2 dz = \frac{1}{2} \cdot \left(\frac{1}{3} z^3 \Big|_0^1 \right) = \frac{1}{2} \left(\frac{1}{3} - 0 \right)$$

$$= \frac{1}{2} \cdot \frac{1}{3}$$

✓ (0/10)

$\boxed{\frac{1}{6}}$

The Answer of Problem 3. (c)

3. (c). Let $I = [0,1]^2$. & $Q = \mathbb{Q} \cap [0,1]$. for $x \in Q$, let $x = \frac{p}{q}$ in lowest term.

define $S(x) = \{(\frac{m}{q}, \frac{n}{q}) : m=0,1,\dots,p, n=0,1,\dots,p\}$

define SCI by $S = \bigcup_{x \in Q} S(x)$ & $f: I \rightarrow \mathbb{R}$ by $f(x,y) = \begin{cases} 0 & \text{if } (x,y) \in S \\ 1 & \text{if } (x,y) \notin S \end{cases}$

So. $\int_0^1 \int_0^1 f(x,y) dx dy = \int_0^1 \int_0^1 f(x,y) dx dy = 1$ (both exist)

But f is not integrable on $I = [0,1]^2$

Proof: I. Show $\int_0^1 \int_0^1 f(x,y) dx dy$ exists.

Consider $\int_0^1 f(x,y) dy$. by fixed x , $x \in [0,1]$. can be Q or not Q .

① if $x \in Q$, when $y \in \left\{ \frac{n}{q} \mid n=0,1,\dots,p \right\}$, then $f(x,y)=0$, since y is countable ($(p+1)$) in this case
& otherwise $f(x,y)=1$.

Since $f(x,y)=0$ is countable. & $f(x,y)=1$ is infinite, we can find a partition set

$$\int_0^1 f(x,y) dy = \int_0^1 f(x,y) dy = \int_0^1 f(x,y) dy = \int_0^1 1 dy = 1$$

② if $x \notin Q$, then whatever the y is, $f(x,y)=1$. then $\int_0^1 f(x,y) dy = \int_0^1 1 dy = 1$

Hence $\int_0^1 f(x,y) dy$ exists & equals to 1. & then $\int_0^1 \left(\int_0^1 f(x,y) dy \right) dx = \int_0^1 1 dx = 1$ exist.

II. Since the function is symmetric, so. prove $\int_0^1 \left(\int_0^1 f(x,y) dx \right) dy$ is same & equals 1.

III. Show f is not integrable on $I = [0,1]^2$.

① P be an arbitrary Partition of I. for tiny REI, we can find at least two rational # & two irrational #. So. when x,y are both rational. $f(x,y)=0 = m_R(f)$.

If one is irrational then $f(x,y)=1 = M_R(f)$.

$$\text{So. } U(f,P) = \sum_{R \in P} M_R(f) \cdot \lambda(R) = 1 \quad L(f,P) = \sum_{R \in P} m_R(f) \cdot \lambda(R) = 0.$$

Let $\epsilon = \frac{1}{2}$. whatever how to refine the partition, $U(f,P) - L(f,P) = 1 - 0 = 1 > \frac{1}{2} = \epsilon$

so. f is not integrable on I.

Therefore $\int_0^1 \left(\int_0^1 f(x,y) dx \right) dy$ & $\int_0^1 \left(\int_0^1 f(x,y) dy \right) dx$ exists but. f is not integrable on I

② $\int_{x \in A} \int_{y \in B} f(x,y) dy dx$ & $\int_{y \in B} \int_{x \in A} f(x,y) dx dy$ exist, but $\int_I f(x,y) dx dy$ not exist.

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