Wed 1st Feb Hour: 048

Today: Some multilinear algebra
Read along: sec 26-27

Def: $\mathcal{L}^k(V) := \{ F: V^k \to \mathbb{R} : \forall 1 \le i \le k,\ \forall v_1, \cdots, \hat{v_i}, \cdots v_k \in V$ (k-1 vectors), $v \mapsto F(v_1, \cdots, v_{i-1}, v, v_{i+1}, \cdots v_k)$ is linear $\}$

Also called k-linear forms, linear in each variable seperately.

Claim: $\mathcal{L}^k(V)$ is a vector space.
Better phrasing: $\mathcal{L}^k(V)$ is a sub-vector space of Function $(V^k \to \mathbb{R})$
(Aside: a set X: {Function $X \to \mathbb{R}$} is a vector space)

NTS: if $f, g \in \mathcal{L}^k(V)$, $f+g \in \mathcal{L}^k(V)$
if $\lambda \in \mathbb{R}$, $f \in \mathcal{L}^k(V)$, $\lambda f \in \mathcal{L}^k(V)$

f.g: $(f+g)(v+v', v_2, \cdots, v_k) = f(v+v', v_2, \cdots, v_k) + g(v+v', v_2, \cdots, v_k)$

$\overset{\text{using}}{\underset{f.g \in \mathcal{L}^k}{=}} f(v, v_2, \cdots, v_k) + f(v', v_2, \cdots, v_k) + g(v, v_2, \cdots, v_k) + g(v', v_2, \cdots, v_k)$

$\overset{\text{def of}}{\underset{f+g.x2}{=}} (f+g)(v, \cdots, v_2, v_k) + (f+g)(v', \cdots, v_2, v_k)$

Thm: If $(a_1, \cdots, a_n)$ is a basis of V, and $I = (i_1, \cdots, i_k) \in \{1, \cdots, n\}^k = \underline{n}^k$, (list of indices)
(where $\underline{n}^k$ is the list of basis vectors, which is index)
Then, there is a unique $\phi_I \in \mathcal{L}^k(V)$ s.t. $\phi_I = (a_{j_1}, \cdots, a_{j_k}) = \begin{cases} 1, & \text{if } I = J = (j_1, \cdots, j_k) \\ 0, & \text{if } I \ne J \end{cases} = \delta_{IJ}$.

Furthermore, $\{\phi_I : I \in \underline{n}^k\}$ is a basis of $\mathcal{L}^k(V)$ (dim $= n^k$)
# of sequences of length k, each member is from 1 to n. so n^k

Pf: Step 1. Lemma: an element of $\mathcal{L}^k(V)$ is determined by its values on sequences of basis vectors.

$\Rightarrow$ If ϕ_1, ϕ_2 both $\in \mathcal{L}^k$, have same values on sequences of basis vector, then $\phi_1 = \phi_2$, so ϕI is uniquely determined

Pf of Lemma: Suppose $\phi \in \mathcal{L}^k$, and $\begin{cases} v_1 = \alpha_{11}a_1 + \alpha_{12}a_2 + \cdots + \alpha_{1n}a_n \\ \vdots \\ v_k = \alpha_{k1}a_1 + \cdots\cdots + \alpha_{kn}a_n \end{cases}$

Then $\phi(v_1, \cdots, v_k) = \phi\left(\sum_{j_1=1}^{n} \alpha_{1j_1} a_{j_1}, \sum_{j_2=1}^{n} \alpha_{2j_2} a_{j_2}, \cdots\right)$

$= \sum_{j_1=1}^{n} \alpha_{1j_1} \cdot \sum_{j_2=1}^{n} \alpha_{2j_2} \cdots \phi(a_{j_1}, a_{j_2}, \cdots, a_{j_k})$ $\square$

Step 2: existence of ϕI (P.S. V always finite dim.)

$\mathcal{L}^1(V) = \{\phi : V \to \mathbb{R} \text{ linear}\} =: V^*$, 'the dual of V'

[Aside: why is $(V^*)^*$ "$=$" V?

There is a unique $\phi_i \in V^*$ s.t. $\phi_i(a_j) = \delta_{ij}$ (already known)]

If $I = (i_1, \cdots, i_k)$, let $\phi I(v_1, \cdots, v_k) = \phi_{i_1}(v_1)\, \phi_{i_2}(v_2) \cdots \phi_{i_k}(v_k)$

a. ϕI is multi-linear.

b. $\phi I(aJ) = \phi_{i_1}(a_{j_1})\, \phi_{i_2}(a_{j_2}) \cdots \phi_{i_k}(a_{j_k})$

$= \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_k j_k} = \delta_{IJ}$

Step 3: $\{\phi I\}$ is linearly independent: assume $0 = \sum_{I \in n^k} \alpha I \phi I$ (*)

Evaluate (*) on $aJ = (a_{j_1}, \cdots, a_{j_k})$

$0 = \sum_I \alpha I \phi I(aJ) = \sum_I \delta_{IJ} \alpha I = \alpha J$. $\square$