

Nov 25.

Read along. 35 43 45

Goal: Tietze, compactness in metric theorem

Correction: X is $T_{3.5}$ iff it is T_4 & $\{[f \neq 0]\}$ is a basis for its topology.

Tietze's existence theorem: If X is T_4 , $A \subset X$ is closed,

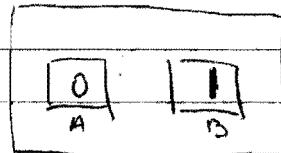
and $f: A \rightarrow \mathbb{R}$ is continuous, then $\exists \tilde{f}: X \rightarrow \mathbb{R}$ cont. s.t. $\tilde{f}|_A = f$

Claim: Tietze \Rightarrow Urysohn. (A, B closed & $A \cap B = \emptyset$)

$$C = A \cup B$$

$$f: C \rightarrow \mathbb{R}$$

$$\exists \tilde{f}: X \rightarrow \mathbb{R}$$



2.g , 5th

$$+ \tilde{f} + + +$$

proof: Assume first that f is bounded

Lemma: If $f: C \rightarrow \mathbb{R}$ is cont. & $\tilde{f}_0: X \rightarrow \mathbb{R}$ s.t. $|f - \tilde{f}_0| \leq \varepsilon$ on C .

$$|f - \tilde{f}| \leq \frac{2}{3}\varepsilon$$

on C

then $\exists \tilde{f}_1: X \rightarrow \mathbb{R}$ s.t. $|\tilde{f}_0 - \tilde{f}_1| \leq \varepsilon$ everywhere &

$$|f - \tilde{f}| \leq \frac{2}{3}\varepsilon \quad \text{can improve the guess.}$$

Pf: Let $A = \{x \in C \mid f(x) - \tilde{f}_0(x) \geq \frac{1}{3}\varepsilon\}$. C is closed

$$B = \{x \in C \mid f(x) - \tilde{f}_0(x) \leq -\frac{1}{3}\varepsilon\}$$

then A & B are closed, so by Urysohn, find $g: X \rightarrow [-\frac{1}{3}\varepsilon, \frac{1}{3}\varepsilon]$

$$\text{s.t. } \forall a \in A, g(a) = \frac{1}{3}\varepsilon, \forall b \in B, g(b) = -\frac{1}{3}\varepsilon$$

$$\text{set } \tilde{f}_1 = \tilde{f}_0 + g, |\tilde{f}_0 - \tilde{f}_1| = |g| \leq \frac{\varepsilon}{3} < \varepsilon.$$

$$|f(x) - \tilde{f}_1(x)| = |f(x) - \tilde{f}_0(x) - g(x)| =$$

for $x \in C$

$$= \begin{cases} \frac{1}{3}\varepsilon \leq I \leq \varepsilon, II = \frac{1}{3}\varepsilon. \Rightarrow 0 \leq I - II \leq \frac{2}{3}\varepsilon & x \in A \\ -\frac{2}{3}\varepsilon \leq I - II \leq 0 & x \in B \\ -\frac{1}{3}\varepsilon < I < \frac{1}{3}\varepsilon, |II| \leq \frac{1}{3}\varepsilon \Rightarrow -\frac{2}{3}\varepsilon < I - II < \frac{2}{3}\varepsilon & x \in C - A - B \end{cases}$$

$$\Rightarrow |f(x) - \tilde{f}_1(x)| \leq \frac{2}{3}\varepsilon. \quad \square \text{ (lemma)}$$

start with $\tilde{f}_0 \equiv 0$ $|f - \tilde{f}_0| \leq M$ on C . (for some M)

use the lemma to construct $\tilde{f}_1, \tilde{f}_2, \dots$ s.t. $|f - \tilde{f}_n| \leq (\frac{2}{3})^n M$ on C .

and $|\tilde{f}_n - \tilde{f}_{n+1}| \leq (\frac{2}{3})^n M$ on X

Goal: Set $\tilde{f}(x) = \lim_{n \rightarrow \infty} \tilde{f}_n(x)$ check: ① \tilde{f}_n converges ② continuous

check: ① (\tilde{f}_n) converges. B/c it is a cauchy sequence.

Reminder: A sequence (x_n) is "cauchy" if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq N$.

$\exists N$. s.t. for $\forall n, m > N$. $d(x_n, x_m) < \epsilon$

Definition: T is "complete" if every cauchy seq in T converges.

Thm: \mathbb{R} is complete

Example: \mathbb{Q} is not complete

$$\hat{f}(x) = f(x) \text{ on } C \text{ as. } |f(x) - \hat{f}_n(x)| \rightarrow 0$$

e.g.

continuity: Note: the lim of a seq of cts function may not be cts.

Def: We say that a seq $f_n: X \rightarrow \mathbb{R}$ "converges uniformly" to $f: X \rightarrow \mathbb{R}$ if $\forall \epsilon > 0 \exists N$, s.t. $|f_n(x) - f(x)| < \epsilon$ for $\forall n > N$ $\forall x \in X$

Lemma: If $f_n: X \rightarrow \mathbb{R}$ cont, & $f_n \xrightarrow{\text{unif}} f$, then f is cont

Pf: Given $x_0 \in X$ & $\epsilon > 0$, find n large enough s.t.

$$|f_n - f| < \frac{\epsilon}{3} \text{ everywhere}$$

by continuity of f_n , find a nbd U of x_0 s.t.

$$x \in U, |f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}$$

Now for any $x \in U$,

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \boxed{\text{Pf.}} \end{aligned}$$

Unbounded Tietze.

Now assume $f: C \rightarrow \mathbb{R}$ is not necessarily bdd.

let $g: \mathbb{R} \rightarrow (-1, 1)$ be a homeomorphism.

(e.g. $g(x) = \arctan x$)

Then $g \circ f: C \rightarrow (-1, 1)$ is a cont. func on C . find

an extension $\widehat{g \circ f}: X \rightarrow \mathbb{R}$ (goal: set $\widehat{f} = g^{-1} \circ \widehat{g \circ f}$)

Let $B = \widehat{g \circ f}^{-1}(\mathbb{R} - (-1, 1))$. then B is closed & $B \cap C = \emptyset$

By Urysohn, find $h: X \xrightarrow{\text{multiply}} [0, 1]$ s.t. $h|_C = 1$, $h|_B = 0$

consider the function $h \cdot \widehat{g \circ f}$, its range $\subset (-1, 1)$.

so $\widehat{f} = h \cdot \widehat{g \circ f}$ is well defined & cts

if $x \in C$, $\widehat{f}(x) = g^{-1}(h(x) \cdot \widehat{g \circ f}(x)) = g^{-1}(1 \cdot g \circ f(x)) = f(x)$