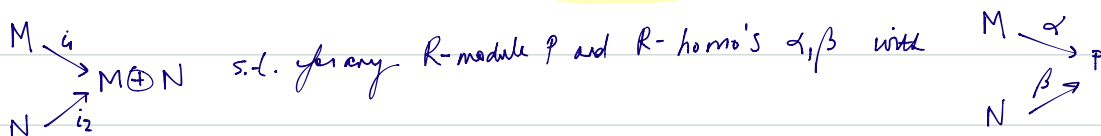
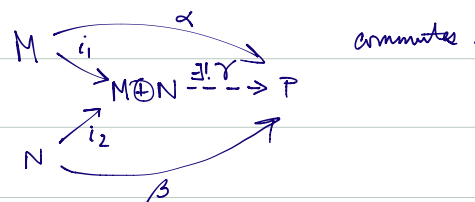


Defⁿ: Let M and N be R -modules. A direct sum of M and N , $M \oplus N$, is a diagram



$\exists! \gamma: M \oplus N \rightarrow P$, R -mod. homo. s.t.

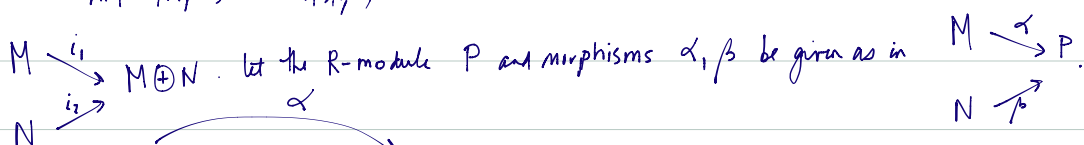


Thm 1. There exists a direct sum of M and N , and it is unique up to isomorphism.
 \hookrightarrow I.e. all models of $M \oplus N$ are isomorphic.

Thm 2. i_1, i_2 in the defⁿ of $M \oplus N$ are injective.

Proof of Thm 1. Let $M \oplus N$ be defined as the set $(m, n) \in M \times N$ to + and \cdot given by $(m_1, n_1) + (m_2, n_2) := (m_1 + m_2, n_1 + n_2)$; $\forall r \in R, r(m, n) := (rm, rn)$. It's clear this defines an R -module.

Let $i_1: M \rightarrow M \oplus N$; $i_2: N \rightarrow M \oplus N$ $m \mapsto (m, 0)$ $n \mapsto (0, n)$ (Note: this is the standard model of a direct sum of $M \oplus N$.)



So, $\begin{array}{ccc} M & \xrightarrow{i_1} & M \oplus N \\ N & \xrightarrow{i_2} & \end{array} \xrightarrow{\exists! \gamma} P$ We show that there exists a unique R -mod. morphism γ s.t. this diagram commutes. Indeed, define γ by $\gamma(m, n) := \alpha(m) + \beta(n)$.

NTS: $\alpha = \gamma \circ i_1, \beta = \gamma \circ i_2$. But $\alpha(m) = \gamma(i_1(m)) = \gamma(m, 0) := \alpha(m) + \beta(0) = \alpha(m)$. Sim., $\beta(n) = \gamma(i_2(n)) = \gamma(0, n) = \alpha(0) + \beta(n) = \beta(n)$.

Now, $\gamma(m_1, n_1) + \gamma(m_2, n_2) = \alpha(m_1) + \beta(n_1) + \alpha(m_2) + \beta(n_2) = \alpha(m_1 + m_2) + \beta(n_1 + n_2) = \gamma(m_1 + m_2, n_1 + n_2)$
 and $r\gamma(m, n) = r[\alpha(m) + \beta(n)] = r\alpha(m) + r\beta(n) = \alpha(rm) + \beta(rn) = \gamma(rm, rn) = \gamma(r(m, n))$.

$\therefore \gamma$ is an R -mod. morphism.

Next we show that γ is unique. Suppose γ' is an R -mod. morphism s.t. $\alpha = \gamma' \circ i_1, \beta = \gamma' \circ i_2$.

Then, $\alpha(m) = \gamma' \circ i_1(m) = \gamma'(m, 0)$; $\beta(n) = \gamma' \circ i_2(n) = \gamma'(0, n)$. But $\gamma'(m, n) = \gamma'(m, 0) + \gamma'(0, n) = \alpha(m) + \beta(n) = \gamma(m, n)$. $\therefore \gamma' = \gamma$. So, this model of a direct sum is indeed an $M \oplus N$.

Next, we show that all models of $M \oplus N$ are isomorphic. For this, it suffices to show that all