December 1st, 2009

MAT240: Abstract Linear Algebra Lecture:

Theorem: If $det` = M_{nxn} \rightarrow F$ satisfies 0-3, then $det` = det$

Theorem: If $A = E_1 \ldots E_n$ is a product of elementary matrices and B is a matrix in RREF, then

$$\det(A) = \det(E_1) * \det(E_2) \ldots \det(E_n)\, B$$

Proof: $\det(A) = \det\big(E_1(E_2 \ldots E_n B)\big) = \det(E) * \det(A`) * \det(B)$

$$= \det(E_1) * \det(E_2 \ldots E_n B) = \det(E_1) * \det(E_2) * \ldots * \det(E_n) * \det(B)$$

Theorem: A is invertible $\leftrightarrow \det(A) \neq 0 \rightarrow \det(A)$ is invertible.

Proof: Write $A = E_1 \ldots E_n B$ where B is in RREF

A is invertible $\overset{by\ following\ \ lemma}{\Longleftrightarrow} E_1 \ldots E_n$ & B are invertible.

$\leftrightarrow \det(E_1) \neq, and \det(E_2) \neq 0\ and \ldots and \det(B) \neq 0$

$\leftrightarrow\ |E_1||E_2| \ldots |B| \leftrightarrow \det(A) = \det(E_1 \ldots E_n B) \neq 0$

Lemma: If A, B are nxn square matrices $(A, B \in M_{nxn})$ then AB is invertible iff both AB are invertible. Likewise $E_1 \ldots E_n$ is invertible iff each E_i is invertible.

Comment: If B is RREF:

B has a row of 0's	B has no row of 0's
• det(B) = 0 • B is not invertible	• det(B) = 1 • B is invertible

Proof of Lemma: Suppose A and B are invertible:

$$AB(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})AB = \cdots = I$$

$\rightarrow AB$ is invertible

Assume (AB) is invertible, $\exists\, C\ s.t. (AB)C = C(AB) = I$

Claim: BC is an inverse of A (so A is invertible)

Proof: $ABC = (AB)C = I$ and by a theorem that we've proven, if a square matrix has a right inverse, then its right inverse is also a left inverse.

Theorem: $\det(AB) = \det(A)\det(B)$

Proof: Either A or B are is not invertible

e.g if I is not invertible, $\det(A) = 0$ and also, AB is not invertible so $\det(AB) = 0$.

Otherwise, both A&B are invertible, so write $A = E_1 \dots E_n I = E_1 \dots E_n$

and $B = E`_1 \dots E`_m$ so $\det(AB) = \det(E_1 \dots E_n * E`_1 \dots E`_m)$

$= \det(E_1)\det(E_2) \dots \det(E_n)\det(E`_1) \dots \det(E`_m)$

$= \det(E_1 \dots E_n)\det(E`_1 \dots E`_m)$

$= \det(A)\det(B)$

Corollary:

A is invertible iff A^T is $(A^T)^{-1} = (A^{-1})^T$

Proof: Exercise

Theorem: $\det(A^T) = \det(A)$

Proof: If A isn't invertible 0 = 0

Otherwise, write A = $E_1 \dots E_n$

$\det(A^T) = \det((E_1 \dots E_n)^T) = \det\left({E_n}^T {E_{n-1}}^T \dots {E_1}^T\right)$

$= \det\left({E_n}^T\right)\det\left({E_{n-1}}^T\right) \dots \det\left({E_1}^T\right) = \det(E_1)\det(E_2) \dots \det(E_n) = \det(A)$

∴ Moral of $\det(A^T) = \det(A)$ → anything true for rows is also true for columns.