Feb. 10th Fri. hour 052

Today: alternating tensors / dets wedge products / tensor vectors

Read along: 27 - 29

Thm: $\forall I \in \binom{n}{k}, \exists ! \, \psi_I \in A^k(V)$ such that $\forall J \in \binom{n}{k}$, $\psi_I(a_J) = \delta_{IJ}$ and $\{\psi_I\}$ makes a basis, hense $\dim A^k(V) = \binom{n}{k}$

$\mathbb{R}^3$ example: $I = (1,2)$, $\psi_I\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = x_1 \cdot y_2 - x_2 \cdot y_1$

$$= \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$, this is general.

claim: In $V = \mathbb{R}^n$, w/ $a_i = e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \to i$th, given $I \in \binom{n}{k}$,

vectors with length n.

lhs $\psi_I(\vec{x}_1, \cdots, x_k) = \det(X_I)$ rhs, where X_I indicates the matrix formed from rows $(i_1, i_2, \cdots, i_k)$ of the matrix $X_{n \times k}$

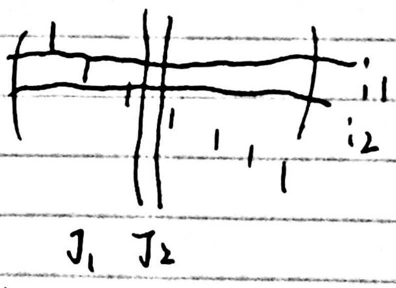
$X_{k \times k} = (x_1 | \cdots | x_k)$

$$\det(X_I) = \det \begin{pmatrix} x_1 \cdot i_1 & x_2 \cdot i_1 & \cdots & x_k \cdot i_1 \\ x_1 \cdot i_2 & \cdots & & x_k \cdot i_2 \\ \vdots & \vdots & & \vdots \\ x_1 \cdot i_k & \cdots & & x_k \cdot i_k \end{pmatrix}$$

i_1 entry of x_1

pf: both sides are alternating and multi-linear in $x_1, \cdots, x_k$. So both sides are determined by their values on a_J, $J \in \binom{n}{k}$.

lhs $(a_J) = \psi_I(a_J) = \delta_{IJ}$

rhs $(a_J) = \det(e_{j_1} | \cdots | e_{j_k})_I = \det$ 

$= \begin{cases} \det I_{k \times k} = 1 & \text{if } I = J \\ \det(\text{some 0-1 matrix with row of 0}) = 0, & \text{if } I \neq J \end{cases}$ $\square$.

example in $\mathbb{R}^3$

$A^0 = \mathrm{Span}(\psi_0)$, $\psi_0() = 1$ $\quad 1$

$A^1 = \mathrm{Span}(\psi_1\ \psi_2\ \psi_3)$, $\psi_i\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_i$ $\quad 3$

$A^2 = \mathrm{span}(\psi_{12}\ \psi_{13}\ \psi_{23})$, $\psi_{13}\begin{pmatrix}x_1 & y_1\\ x_2 & y_2\\ x_3 & y_3\end{pmatrix} = \begin{vmatrix}x_1 & y_1\\ x_3 & y_3\end{vmatrix}$. 3

$\psi_{23}\left(\ \}\ \ \}\ \right) = \begin{vmatrix}x_2 & y_2\\ x_3 & y_3\end{vmatrix}$

$A^3 = \mathrm{Span}(\psi_{123})$. $\psi_{123} = |x|y|z|$. det $\quad 1$

$A^4 = \phi = \{0\}$

$\otimes : \mathcal{L}^k(U) \otimes \mathcal{L}^l(U) \to \mathcal{L}^{k+l}(U)$

$\quad F \qquad g$

$F\otimes g(x_1, \cdots, x_{k+l}) = F(x_1, \cdots, x_k)\cdot g(x_{k+1}, \cdots, x_{k+l})$

suppose $F.g$ are in A^k, A^l. ~~open~~ respectively. Is $F\otimes g = A^{k+l}$?

A: $(\underset{\uparrow A^1}{\psi_1} \otimes \underset{\uparrow A^1}{\psi_2})(x, y) = x_1 . y_2$. No. need a new "$\otimes$"

Def: if $F \in A^k$, $g \in A^l$, then the "wedge". $F\wedge g$ is

$$(F\wedge g)(x_1, \cdots, x_{k+l}) = \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} (F\otimes g)^\sigma (-1)^\sigma$$

normalization factor

$$= \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} F(x_{\sigma_1}, \cdots, x_{\sigma_k})\cdot g(x_{\sigma_{k+1}}, \cdots, x_{\sigma_{k+l}})\cdot(-1)^\sigma$$

$$= \sum_{\sigma\in S_{k+l}} (-1)^\sigma F\sim . g\sim, \text{ where } \begin{matrix}\sigma_1 < \sigma_2 < \cdots < \sigma_k\\ \sigma_{k+1} < \sigma_{k+2} < \cdots < \sigma_{k+l}\end{matrix}$$

Example:

In $A^*(\mathbb{R}^4)$, $(\underbrace{\underset{\psi_{A^2}}{\psi_{13}} \wedge \underset{\psi_{A^1}}{\psi_4}}_{A^3})(x_1, x_2, x_3) = (\psi_{13})(x_1, x_2)\cdot\psi_4(x_3)$

$\qquad - (\psi_{13})(x_1, x_3)\cdot\psi_4(x_2)$

$\qquad + (\psi_{13})(x_2, x_3)\cdot\psi_4(x_1)$

$\sigma\in S_3$, have $\sigma_1 < \sigma_2$, $\sigma_3 \sim$

1 2 3

↓ ↓↓

1 2 3

1 3 2

2 3 1