******************DISCLAIMER***************************
Consult these solutions at your own risk! Perfection of the following answers is not guaranteed or implied, etc etc
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MAT257: Problem Set 17

Section 32

3.
Post-edit: I probably have sign errors here
Let $\omega = xydx + 2zdy - ydz$ and $\alpha(u,v) = (uv, u^2, 3u+v)$. Then

$$\begin{aligned}
d\omega &= d(xydx + 2zdy - ydz) = xdy \wedge dx + 2dz \wedge dy - dy \wedge dz \\
\alpha^*\omega &= (uv)(u^2)d(uv) + 2(3u+v)d(u^2) - (u^2)d(3u+v) \\
&= u^3v(udv + vdu) + (6u+2v)(2udu) - u^2(3du + dv) \\
&= (u^4v - u^2)dv + (u^3v^2 + 9u^2 + 4uv)du \\
\alpha^*(d\omega) &= (uv)d(u^2) \wedge d(uv) + 2d(3u+v) \wedge d(u^2) - d(u^2) \wedge d(3u+v) \\
&= uv(2udu) \wedge (udv + vdu) + 2(3du + dv) \wedge (2udu) - (2udu) \wedge (3du + dv) \\
&= 2u^3vdu \wedge dv + 4udv \wedge du - 2udu \wedge dv \\
&= (2u^3v - 6u)du \wedge dv \\
d(\alpha^*\omega) &= d((u^4v - u^2)dv + (u^3v^2 + 9u^2 + 4uv)du) \\
&= (4u^3v - 2u)du \wedge dv + (2u^3v + 4u)dv \wedge du \\
&= (2u^3v - 6u)du \wedge dv
\end{aligned}$$

5.
Let A be open in $\mathbb{R}^k$ and $\alpha : A \to \mathbb{R}^n$ be of class C^∞. Let $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^n$. Let $I = (i_1, ..., i_l)$ be an ascending l-tuple of the integers from

1 to n.
Since $\alpha^*(dy_I)$ is an l-form, it can be uniquely rewritten as
$\alpha^*(dy_I) = \sum_{[J]} h_J dx_J$
where $J = (j_1, ..., j_l)$ is an ascending l-tuple of integers from 1 to k.
Notice that h_J is the value $\alpha^*(dy_I)$ takes when evaluated on the ascending l-tuple of basis vectors $\{e_{j_1}, \cdots, e_{j_l}\}$. Therefore, as in the proof of Theorem 32.2, we have
$h_J = dy_I(\alpha(x))(\alpha^*(x; e_{j_1}), \cdots, \alpha^*(x; e_{j_l}))$,
$h_J = dy_I(\alpha(x))((\alpha(x); \frac{\delta\alpha}{\delta x_{j_1}}), \cdots, (\alpha(x); \frac{\delta\alpha}{\delta x_{j_l}}))$
$h_J = \det[\frac{\delta\alpha}{\delta x_{j_1}} \cdots \frac{\delta\alpha}{\delta x_{j_l}}]_I = \det \frac{\delta\alpha_I}{\delta x_J}$
Therefore we have
$\alpha^*(dy_I) = \sum_{[J]}(\det \frac{\delta\alpha_I}{\delta x_J}) dx_J$, as required.

Section 33

2.
Let $\alpha : (0,1)^3 \to \mathbb{R}^4$ be given by $\alpha(s,t,u) = (s, u, t, (2u-t)^2)$.
Let $\omega = x_1 dx_1 \wedge dx_4 \wedge dx_3 + 2x_2x_3 dx_1 \wedge dx_2 \wedge dx_3$. Then

$$\begin{aligned}
\int_{Y_\alpha} \omega &= \int_{(0,1)^3} \alpha^*\omega \\
&= \int_{(0,1)^3} s ds \wedge d((2u-t)^2) \wedge dt + 2ut ds \wedge du \wedge dt \\
&= \int_{(0,1)^3} s ds \wedge ((8u-4t)du - (4u-2t)dt) \wedge dt + 2ut ds \wedge du \wedge dt \\
&= 2\int_{(0,1)^3} (4su - 2st + ut) ds \wedge du \wedge dt \\
&= 2\int_0^1 (\int_0^1 (\int_0^1 (4su - 2st + ut) ds) du) dt
\end{aligned}$$

$$= 2\int_0^1 (\int_0^1 (2u - t + ut)du)dt$$
$$= 2\int_0^1 (1 - t + \frac{1}{2}t)dt = \frac{3}{2}$$

3.

a)

Let A be the open unit 1-ball.

Define $\alpha : A \to \mathbb{R}^3$ by $\alpha(u, v) = (u, v, (1 - u^2 - v^2)^{1/2})$.

Let $\omega = \frac{1}{||x||^m}(x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2)$

$$\int_{Y_\alpha} \omega = \int_A \alpha^* \omega$$
$$= \int_A \frac{(udv \wedge \frac{-udu}{\sqrt{1-u^2-v^2}}) - (vdu \wedge \frac{-vdv}{\sqrt{1-u^2-v^2}}) + (1 - u^2 - v^2)^{1/2} du \wedge dv}{(\sqrt{u^2 + v^2 + (1 - u^2 - v^2)})^m}$$
$$= \int_A \frac{-u^2}{\sqrt{1 - u^2 - v^2}} dv \wedge du - \frac{-v^2}{\sqrt{1 - u^2 - v^2}} du \wedge dv + (1 - u^2 - v^2)^{1/2} du \wedge dv$$
$$= \int_A (\frac{u^2}{\sqrt{1 - u^2 - v^2}} + \frac{v^2}{\sqrt{1 - u^2 - v^2}} + \sqrt{1 - u^2 - v^2}) du \wedge dv$$
$$= \int_A \frac{1}{\sqrt{1 - u^2 - v^2}} du \wedge dv$$
$$= \int_0^1 (\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \frac{1}{\sqrt{1 - u^2 - v^2}} du) dv$$
$$= \int_0^1 (\int_0^{2\pi} \frac{r}{\sqrt{1 - r^2}} d\theta) dr$$
$$= 2\pi \int_0^1 \frac{r}{\sqrt{1 - r^2}} dr$$
$$= -2\pi(\sqrt{1 - r^2}|_0^1)$$
$$= 2\pi$$

b)
Define $\alpha : A \to \mathbb{R}^3$ by $\alpha(u,v) = (u, v, -(1-u^2-v^2)^{1/2})$.

$$\int_{Y_\alpha} \omega = \int_A \frac{(udv \wedge \frac{udu}{\sqrt{1-u^2-v^2}}) - (vdu \wedge \frac{vdv}{\sqrt{1-u^2-v^2}}) - (1-u^2-v^2)^{1/2} du \wedge dv)}{(\sqrt{u^2+v^2+(1-u^2-v^2)})^m}$$

$$= \int_A \frac{-u^2+v^2-(\sqrt{1-u^2-v^2})^2}{\sqrt{1-u^2-v^2}} du \wedge dv$$

$$= \int_A \frac{2v^2-1}{\sqrt{1-u^2-v^2}} du \wedge dv$$

$$= \int_0^1 (\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \frac{2v^2-1}{\sqrt{1-u^2-v^2}} du) dv$$

$$= \int_0^1 (\int_0^{2\pi} \frac{r(2r^2 \sin^2\theta - 1)}{\sqrt{1-r^2}} d\theta) dr$$

$$= \int_0^1 \frac{2r^3}{\sqrt{1-r^2}} (\int_0^{2\pi} (\sin^2\theta - \frac{1}{2r^3}) d\theta) dr$$

$$= \int_0^1 \frac{2r^3}{\sqrt{1-r^2}} (\pi - \frac{\pi}{r^3})) dr$$

$$= \int_0^1 \frac{2\pi(r^3-1)}{\sqrt{1-r^2}} dr$$

$$= 2\pi(-\arcsin(1) + (\frac{2}{3}))$$

$$= \pi(\frac{4}{3} - \pi)$$