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Suppose some $v \in V$ can be written as a l.c. in two ways.

$$\sum_{i=1}^{n} \alpha_i' u_i = v = \sum_{i=1}^{n} \alpha_i u_i \quad u_i \in \beta.$$

then

$$0 = v - v = \sum \alpha_i u_i - \sum \alpha_i' u_i$$

$$= \sum (\alpha_i - \alpha_i') u_i$$

is a l.c. of elements of β equal too

but β is lin. indep $\Rightarrow \forall i, \alpha_i - \alpha_i' = 0$

So $\forall i$ $\alpha_i = \alpha_i'$ So the two l.c. expressing v are in fact the same. Suppose every v can be written in a unique way as $v = \sum \alpha_i u_i$ then β is a basis indeed.

1. Since v is a l.c. of elements of β

$\text{span}(\beta) = V$

2. β is lin indep. Indeed, suppose

$$\sum \alpha_i u_i = 0 = \sum_{i=1}^{n} 0 \cdot u_i$$

$\Rightarrow \forall i \ \alpha_i = 0$

$\square$

Thm If a finite set S generates a V.S. V, then there is some subset $\beta \subset S$ which is a basis (β could be $\beta = S$)

Comment: The same is true for infinite S, but much harder.

Pf. Let m be the maximal size of a subset of S which is lin. indep.

(since S is finite every li. subset of S is size $\leq |S|$. So $m \leq |S|$)

(ϕ is a subset of S which is lin indep, so $m \geq 0$).

Let β be some lin. indep subset of S s.t. $|\beta| = m$. I claim that β is a basis

1. β is lin. indep by choice

2. Let $v \in S$. If $v \in \beta$ then $v \in \operatorname{span}(\beta)$

Otherwise $v \notin \beta$

<u>Thm</u> If a v.s. V has a finite basis, then every other basis of V is also finite and has the same number of elements.

<u>Def</u> If V has a finite basis,

We say that V is finite-dimensional and set $\dim V := \left(\begin{array}{l}\text{The number of elements}\\ \text{in (any) basis of } V\end{array}\right)$

Example 1. $\dim F^n = n$

take $$\beta = \underbrace{\left(\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right)}_{n}$$