Homework 10. TUT: R 4-5 P.M.

Section 15: 1. ① show $\lim_{N\to\infty}\int_{C_N} f = \lambda$. for a sequence C_N of compact rectifiable subset of $\mathbb{R}$ s.t $C_N \uparrow \mathbb{R}$.

let $C_N = [a_N, b_N]$ s.t $a_N = -\sqrt{N|2\lambda|}$, $b_N = \sqrt{a_N^2 + 2\lambda}$ since $a_N^2 = N|2\lambda| \geq |2\lambda| \Rightarrow a_N^2 + 2\lambda \geq 0$

so. b_N make sense. & $b_N \geq 0$, $a_N \leq 0$, C_N is a general interval $\Rightarrow C_N$ compact & rectifiable.

as N increasing, a_N decreasing & b_N increasing. ⓘⓔ $a_{N+1} \leq a_N$ & $b_N \leq b_{N+1}$ $N = 1, 2, \ldots$

Hence $C_N \subset \text{int } C_{N+1}$, $C_N \uparrow \mathbb{R}$ ⓘⓔ $\bigcup C_N = \mathbb{R}$.

Since $f(x) = x$ is cont. on compact set $C_N \Rightarrow f$ is uniformly cont. on $C_N \Rightarrow f$ is integrable on C_N

$N|2\lambda| \geq 2\lambda$

Then $\lim_{N\to\infty}\int_{C_N} f$ exists. & $\lim_{N\to\infty}\int_{C_N} f = \lim_{N\to\infty}\int_{[a_N, b_N]} x = \lim_{N\to\infty}\frac{1}{2}x^2\Big|_{a_N}^{b_N} = \lim_{N\to\infty}\frac{1}{2}[(b_N)^2 - (a_N)^2]$

$= \lim_{N\to\infty}\frac{1}{2}[(a_N^2 + 2\lambda) - a_N^2] = \lim_{N\to\infty}\frac{1}{2}(2\lambda) = \boxed{\lambda}$ as required.

② $\text{ext}\int_{\mathbb{R}} f$ NOT exist.

from ①. for an arbitrary $\lambda \in \mathbb{R}$, we can find a sequence C_N of compact rectifiable subsets of $\mathbb{R}$, s.t. $\bigcup C_N = \mathbb{R}$ & $C_N \subset \text{int } C_{N+1}$ for each N. & $\lim_{N\to\infty}\int_{C_N} f = \lambda$.

Then $\int_{C_N} f$ is not bounded. since we are make $\lambda \to \infty$. ⓘⓔ λ is not bounded

since $\lim_{N\to\infty}\int_{C_N} f = \lim_{N\to\infty}\boxed{\int_{C_N}}(f_+ - f_-) \leq \lim_{N\to\infty}\int_{C_N} f_+ + f_- = \lim_{N\to\infty}\int_{C_N}|f|$, f_+, f_- are non-negative

Hence. $\int_{C_N}|f|$ is not bounded. $\Rightarrow \text{ext}\int_{\mathbb{R}} f$ not exist.

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3. a). $A = \{(x,y) \mid x > 1 \text{ \& } y > 1\}$. & $f(x,y) = \frac{1}{\sqrt{xy}}$

$B = (0,1)^2$

① claim: $\boxed{\text{ext}\int_A f \text{ NOT exist}}$

proof: f is bounded on A. since $0 < f < 1 \Rightarrow |f| = f$, A is not bounded

Setting $U_N = (1, N)^2$ then $U_1 \subset U_2 \subset \cdots$ is a sequence of open sets whose union is A.

The set U_N is rectifiable. since U_N is bounded & $\text{bd } U_N = \{1\}\times[1,N] \cup \{N\}\times[1,N]$ has measure-0 in $\mathbb{R}^2$. & $\overline{U_N} = [1,N]^2$.

f is bounded on U_N. since $\overline{U_N}$ is compact & f is cont. on $\overline{U_N}$

$\Rightarrow \int_{U_N} f$ exists as an ordinary integral, so we can apply the Fubini THM

Compute. $\int_{U_N} f = \int_{x=1}^{x=N}\int_{y=1}^{y=N}\frac{1}{\sqrt{xy}} = \int_{x=1}^{x=N}\frac{1}{\sqrt{x}}(2\sqrt{y}\Big|_1^N) = (2\sqrt{N} - 2)\int_{x=1}^{x=N}\frac{1}{\sqrt{x}} = [2(\sqrt{N}-1)]\cdot 2\sqrt{x}\Big|_1^N$

$= [2(\sqrt{N}-1)]^2 = \boxed{4(\sqrt{N}-1)^2} \to \infty$ as $N \to \infty$.

so. $\int_{U_N}|f| = \int_{U_N} f$ is not bounded. $\Rightarrow \text{ext}\int_A f$ does NOT Exist.

by THM 15.6.

② claim: $\boxed{\text{ext}\int_B f \text{ exists. \& } \text{ext}\int_B f = 4}$

proof: B is bounded by $[0,1]^2$ but f is not bounded on B, since $1 < f \Rightarrow |f| = f$

Indeed, f is unbounded near each point of the x & y axes.

Setting $U_N = (\frac{1}{N}, 1)^2$, then $U_1 \subset U_2 \subset \cdots$ is a sequence of open sets whose union is B.

f is bounded on U_N since $1 < f < \frac{1}{\sqrt{N}\sqrt{N}} = \frac{1}{N}$ on U_N. f is also cont. on U_N.
Then f is integrable on U_N. we can apply the Fubini THM to compute ordinary integral on U_N

$$\int_{U_N} f = \int_{x=1/N}^{x=1} \int_{y=1/N}^{y=1} \frac{1}{\sqrt{xy}} = \int_{x=1/N}^{x=1} \frac{1}{\sqrt{x}} (2\sqrt{y}\Big|_{1/N}^{1}) = (2 - \frac{2}{\sqrt{N}}) \int_{x=1/N}^{x=1} \frac{1}{\sqrt{x}} = [2(1-\frac{1}{\sqrt{N}})] \cdot (2\sqrt{x}\Big|_{1/N}^{1})$$

$$= [2(1-\frac{1}{\sqrt{N}})]^2 = \boxed{4(1-\frac{1}{\sqrt{N}})^2}$$

$\frac{1}{\sqrt{N}} \to 0$ as $N \to \infty$, $\Rightarrow (1-\frac{1}{\sqrt{N}}) \to 1$ as $N \to \infty \Rightarrow (1-\frac{1}{\sqrt{N}})^2 \to 1$ as $N \to \infty$

so $\int_{U_N} f = 4(1-\frac{1}{\sqrt{N}})^2 \to 4$ as $N \to \infty$. (i.e.) $\int_{U_N} f$ bounded

Hence $\int_{U_N} |f| = \int_{U_N} f$ exists & bounded $\overset{\text{THM 15.6}}{\Rightarrow}$ $\boxed{\int_B f = \lim_{N\to\infty} \int_{U_N} f = 4 \text{ (i.e.) } \int_B f \text{ exists}}$

b) want to show $\int_C f$ exists.

$f(x,y) = 1/(x^2+\sqrt{x})(y^2+\sqrt{y})$, $C = \{(x,y) \mid x>0 \ \& \ y>0\}$ $\quad f>0 \Rightarrow |f| = f$.

① let $\boxed{A = \{(x,y) \mid x>1 \ \& \ y>1\}}$. let $g = 1/x^2y^2$

Since $x^2 \le x^2+\sqrt{x}$, & $y^2 \le y^2+\sqrt{y}$ so $x^2y^2 \le (x^2+\sqrt{x})(y^2+\sqrt{y}) \Rightarrow g \ge x$

From textbook Example 2 on Page 131, $\exists$ sequence C_{N_A} of compact rectifiable subsets of A s.t. $\bigcup C_{N_A} = A$.

we know that $\int_A g$ exists & $\int_A g = 1$ (i.e.) $\lim_{N\to\infty} \int_{C_{N_A}} |g| = 1 \Rightarrow \int_{C_{N_A}} |g|$ is bounded.

since f is bounded by g on A then $\lim_{N\to\infty} \int_{C_{N_A}} f \le \lim_{N\to\infty} \int_{C_{N_A}} g$

so $\lim_{N_A\to\infty} \int_{C_{N_A}} |f| = \lim_{N_A\to\infty} \int_{C_{N_A}} f \le \lim_{N_A\to\infty} \int_{C_N} |g| \Rightarrow \int_{C_{N_A}} |f|$ is bounded

Hence $\int_A f$ exists.

② let $B = (0,1)$ let $g = 1/(xy)^{1/2} = \frac{1}{\sqrt{xy}}$

Since $\sqrt{x} \le x^2+\sqrt{x}$ & $\sqrt{y} \le y^2+\sqrt{y}$ so $\sqrt{xy} \le (x^2+\sqrt{x})(y^2+\sqrt{y}) \Rightarrow g \ge x$

From part (a) ②, we have proven that $\int_B g$ exists & $\int_B g = 4 = \lim_{N_B\to\infty} \int_{C_{N_B}} g \Rightarrow \int_{C_{N_B}} |g|$ is bounded

since f is bounded by g on B then $\lim_{N_B\to\infty} \int_{C_{N_B}} f \le \lim_{N_B\to\infty} \int_{C_{N_B}} g$ $\quad$ $\exists$ a sequence C_{N_B} of compact rectifiable subsets of B, $\bigcup C_{N_B} = B$

so $\lim_{N_B\to\infty} \int_{C_{N_B}} |f| = \lim_{N_B\to\infty} \int_{C_{N_B}} f \le \lim_{N_B\to\infty} \int_{C_{N_B}} |g| \Rightarrow \int_{C_{N_B}} |f|$ is bounded.

Hence $\int_B f$ exists

③ let $D = \{(1,1)\}$ then $f(x,y) = \frac{1}{(1^2+\sqrt{1})^2} = \frac{1}{4}$ $\quad \int_D f = 0$

since $C = A \cup B \cup D$ & $A \cap B = \emptyset$, $B \cap D = \emptyset$, $A \cap D$ $\emptyset$.

so $\int_C f = \int_A f + \int_B f + \int_D f$ $\quad$ if $\int_C f$ exists $\Leftrightarrow$ if $\lim_{N\to\infty} \int_{C_N} |f|$ exists $\Leftrightarrow$ if $\int_{C_N} |f|$ bounded

since $\lim_{N\to\infty} \int_{C_N} f = \lim_{N\to\infty} \int_{C_{N_A}} f + \lim_{N\to\infty} \int_{C_{N_B}} f + 0$.

& $\int_{C_{N_A}} f$ is bounded & $\int_{C_{N_B}}$ is bounded $\Rightarrow \int_{C_N} f$ is bounded (i.e.) $\int_{C_N} |f|$ is bounded since $f>0$.

Hence by THM, $\boxed{\int_C f \text{ exists}}$

6. Claim: $\int_A \frac{1}{x\sqrt{y}}$ exists.

proof: let $U_N = (1,N) \times (0, \frac{1}{x}) = \{(x,y) \mid x \in (1,N) \ \& \ y \in (0,\frac{1}{x})\}$ s.t. $U_1 \subset U_2 \subset \cdots$ be a sequence of open sets whose union is $A = \{(x,y) \mid x>1 \ \& \ 0<y<\frac{1}{x}\}$

we try to integrate f over U_N. Let $f(x,y)=\frac{1}{x\sqrt{y}}$.
The set U_N is rectifiable since U_N is bounded by $[1,N]\times[0,1]$ & $Bd\,U_N$ has measure-0 on $\mathbb{R}^2$.
f is bounded on U_N because $\overline{U_N}=[1,N]\times[0,1]$ is compact & f is cont on $\overline{U_N}$.
Thus $\int_{U_N} f$ exists as an ordinary integral, so we can apply the Fubini THM.
compute $\int_{U_N} f=\int_{x=1}^{x=N}\int_{y=0}^{y=1/x}\frac{1}{x\sqrt{y}}=\int_{x=1}^{x=N}\frac{1}{x}\int_{y=0}^{y=1/x}\frac{1}{\sqrt{y}}=\int_{x=1}^{x=N}\frac{1}{x}\left(2\sqrt{y}\Big|_0^{1/x}\right)=\int_{x=1}^{x=N}\frac{1}{x}\left(\frac{2}{\sqrt{x}}-0\right)$

$$=2\int_{x=1}^{x=N}\frac{1}{x\sqrt{x}}=2\int_{x=1}^{x=N}(x)^{-3/2}=2\cdot(-2)x^{-1/2}\Big|_1^N=-4\left(\frac{1}{\sqrt{N}}-\frac{1}{\sqrt{1}}\right)$$

$$=\boxed{-4\left(\frac{1}{\sqrt{N}}-1\right)}$$

as $N\to\infty,\Rightarrow\sqrt{N}\to\infty\Rightarrow\frac{1}{\sqrt{N}}\to0\Rightarrow\frac{1}{\sqrt{N}}-1\to-1\Rightarrow-4\left(\frac{1}{\sqrt{N}}-1\right)\to4$.
So $\lim_{N\to\infty}\int_{U_N} f=4$ (ie) $\int_{U_N} f$ is bounded.
Since $f>0$, $\Rightarrow\int_{U_N} f=\int_{U_N}|f|$ is bounded.
Then By the THM 15.6. $\int_A f$ exists & $\boxed{\int_A f=\lim_{N\to\infty}\int_{U_N} f=4}$

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problem A:

$f:[0,+\infty)\to\mathbb{R}$ $f(x)=\sin(x^2)$

a). prove $\lim_{x\to\infty}f(x)$ does not exist.

proof: prove by contradiction.

Assume the limit exists & $=L$ so $L\in[-1,1]$. since $\sin\in[-1,1]$.
fix. $\varepsilon=1/3$. There exists $M>0$ s.t. for $\forall x>M$, $|f(x)-L|=|\sin(x^2)-L|<\varepsilon=1/3$
we set $\sqrt{P}=\lceil\max\{M,L\}\rceil$ (ie) the ceiling of $\max\{M,L\}$ & $\sqrt{P}\in\mathbb{Z}^+\Rightarrow P\in\mathbb{Z}^+$
There are 2 cases.

① if $L\in[-1,0]$
Let: $x_0=\sqrt{\frac{\pi}{2}+2\pi P}>\sqrt{P}>M\Rightarrow x_0>M$.
Then $|f(x_0)-L|=|\sin(x_0^2)-L|=|\sin(\frac{\pi}{2}+2\pi P)-L|=|\sin(\frac{\pi}{2})-L|=|1-L|$
since $L\in[-1,0]$ so $|1-L|=1-L\geq1>1/3$.
Hence $|f(x_0)-L|>\varepsilon$ contradiction.

② if $L\in(0,1]$
Let $x_0=\sqrt{\frac{3\pi}{2}+2\pi P}>\sqrt{P}>M\Rightarrow x_0>M$
Then $|f(x_0)-L|=|\sin(x_0^2)-L|=|\sin(\frac{3\pi}{2}+2\pi P)-L|=|\sin(\frac{3\pi}{2})-L|=|(-1)-L|$
since $L\in(0,1]$ So $|(-1)-L|=L-(-1)=L+1\geq1>1/3$.
Hence $|f(x_0)-L|>\varepsilon$ contradiction.

Therefore the limit $\lim_{x\to\infty}f(x)$ does not exist.

b). prove $\lim_{N\to\infty}\int_0^N f(x)\,dx$ does exist

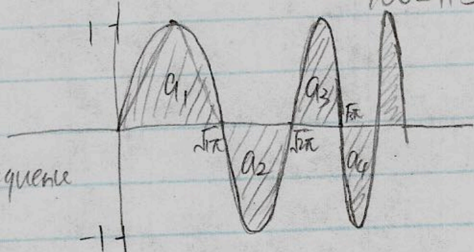
proof: $\lim_{N\to\infty}\int_0^N f(x)\,dx=\lim_{N\to\infty}\int_0^N f_+-f_-$ where $f_+=\max\{f(x),0\}$ $f_-=\max\{-f(x),0\}$

Since $\sin(x^2) \in [-1,1]$ & $\sin(N\pi) = 0$ where $N \in \mathbb{Z}^+$

So let $x_N^2 = N\pi \Rightarrow x_N = \sqrt{N\pi}$

let $a_N = \int_{x=\sqrt{(N-1)\pi}}^{x=\sqrt{N\pi}} \sin(x^2)$ then a_N is a decreasing sequence

s.t. $a_1 > a_2 > a_3 > \cdots > 0$.



since $\sqrt{N\pi} - \sqrt{(N-1)\pi} \to 0$ as $N \to \infty$, & sin is bounded s.t. $\sin \in [-1,1]$

so $\lim_{N\to\infty} a_N = 0$.

Then apply Leibniz's test, then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ converges.

since $\lim_{N\to\infty} \int_0^N f(x)\,dx = a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}$

so $\boxed{\lim_{N\to\infty} \int_0^N f(x)\,dx \text{ converges.}}$ ie $\lim_{N\to\infty} \int_0^N f(x)\,dx$ does exist.

c) proof $\int_{(0,\infty)} f$ does not exist.

<u>proof:</u> let $U_N = (0, \sqrt{N\pi})$ & try to integrate f over U_N.

The set U_N is rectifiable, since U_N is bounded by $[0, \sqrt{N\pi}]$ & the $Bd(U_N)$ has measure-0.

f is bounded on U_N because $\overline{U_N}$ is compact & f is cont. on $\overline{U_N}$.

We compute $\int_{U_N} |f| = \int_{U_N} f_+ + f_- = a_1 + a_2 + a_3 + \cdots + a_N$.

Since $a_N = \int_{x=\sqrt{(N-1)\pi}}^{x=\sqrt{N\pi}} \sin(x^2) \geq \frac{1}{2}(\sqrt{N\pi} - \sqrt{(N-1)\pi}) = \frac{\sqrt{\pi}}{2}(\sqrt{N} - \sqrt{N-1})$ ie f is above the straight line from $(\sqrt{(N-1)\pi}, 0)$ to max point $\Rightarrow a_N \geq$ the Area of the triangle with base $(\sqrt{N\pi} - \sqrt{(N-1)\pi})$ & height 1.

since sin is concave on $[\sqrt{(N-1)\pi}, \sqrt{N\pi}]$.

$$\sum_{i=1}^{N} a_i = a_1 + a_2 + a_3 + \cdots + a_N$$
$$\geq \frac{\sqrt{\pi}}{2}\left[(\sqrt{1} - \sqrt{0}) + (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{N} - \sqrt{N-1})\right]$$
$$\geq \frac{\sqrt{\pi}}{2}(\sqrt{N}) \to \infty \text{ as } N \to \infty.$$

So $\sum_{N=1}^{\infty} a_N = \infty \Rightarrow \lim_{N\to\infty} \int_{U_N} |f| = \sum_{N=1}^{\infty} a_N = \infty$.

Therefore $\int_{U_N} |f|$ is not bounded. $\Rightarrow$ $\int_{(0,\infty)} f$ does NOT exist.

by Thm 15.6.