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MATH401 Homework 2. Tony Yu

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#7. Let R be a finite commutative ring with unity and length n.

let $a \in R$ such that a is not ~~the~~ a unit.

then $a \cdot b \neq 1$ $\forall b \in R$.

since R is finite, we consider

$a \cdot b$ $\forall b \in R$,

~~we then get~~

We have to get a repeat since by the pigeon hole principle, our ring has n-elements and without the unit, we have n-1 units with $a \cdot b_i$ where $i \in 1, 2, \ldots n$.

then $a \cdot b_1 = c$ for some $c \in R$,

$a \cdot b_2 = c$

if $c = 0$, then a is a zero divisor

if $c \neq 0$,

then: $a \cdot b_1 = a \cdot b_2$

$a(b_1 - b_2) = 0$ and therefore, a is a zero divisor.

$\therefore$ all non-zero elements are a unit or a zero divisor.

If we remove the condition ~~that~~ condition that R is finite, consider $\mathbb{Z}$,

which is a finite commutative ring with unity.

But there are ~~also~~ no units and zero-divisors

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Chapter 14.

5. Let $S = \{a+bi \mid a,b \in \mathbb{Z}, b \text{ is even}\}$. Show that S is a subring of $\mathbb{Z}[i]$, but not an ideal

To show this is a subring, we show this is closed under subtraction and multiplication

Let $c,d \in S$ such that: $c = a_1 + b_1 i$ such that $b_{1,2} \in$ even.

$d = a_2 + b_2 i$

$$c-d = a_1 + b_1 i - a_2 - b_2 i$$

$$= \underbrace{a_1 - a_2}_{\in \mathbb{R}} + \underbrace{(b_1 - b_2)}_{\in \text{Even, since } 2k - 2k' = 2(k-k') \in \text{Even.}} i$$

$\in S$.

$$c \cdot d = (a_1 + b_1 i)(a_2 + b_2 i)$$

$$= a_1 a_2 + a_1 \cdot b_2 i + b_1 \cdot a_2 i + b_1 \cdot b_2 i^2$$

$$= \underbrace{(a_1 \cdot a_2 - b_1 \cdot b_2)}_{\in \mathbb{R}} + \underbrace{(a_1 \cdot b_2 + b_1 \cdot a_2)}_{} i$$

$\in$ even since $b_1, b_2 \in$ even, we can factor an ~~even~~ 2 out to show it's even.

$\therefore$ S is a subring.

it is not an ideal since

$i \in \mathbb{Z}[i]$, but:

$i \cdot (a+bi) = -b + ai$

a is not necessarily even.

$\therefore$ S is not ideal.

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11. In the ring of Integer, find a positive integer a such that

a. $\langle a\rangle = \langle 2\rangle + \langle 3\rangle$

$\langle 2\rangle = 2t_1 + 2t_2 + \dots + 2t_n$

$\langle 3\rangle = 3s_1 + 3s_2 + \dots + 3s_m$

$\therefore \langle a\rangle = 2t_1 + 3s_1 + \dots + 2t_n + 3s_m$

$\therefore a = \gcd(2,3)$

$\boxed{a = 1}$

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b) $\langle a\rangle = \langle 3\rangle + \langle 6\rangle$

using a, we know $a = \gcd(3,6)$

$\boxed{a = 3}$

c) $\langle a\rangle = \langle m\rangle + \langle n\rangle$

$a = \gcd(m,n)$

13. Find positive integer such that:

a) $\langle a\rangle = \langle 3\rangle\langle 4\rangle$

$\langle 3\rangle = 3t_1 + \dots + 3t_n$

$\langle 4\rangle = 4s_1 + \dots + 4s_m$

$\therefore \langle a\rangle = \langle 3\rangle\langle 4\rangle$

$= 3t_1 \cdot 4s_1 + 3t_2 \cdot 4s_2 + \dots + 3 \cdot t_n \cdot 4s_m$

$= \langle 12\rangle$

$\therefore a = 12$

b) $\langle a\rangle = \langle 6\rangle\langle 8\rangle$

$a = 48$

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c) $\langle a\rangle = \langle m\rangle\langle n\rangle$

$a = m \cdot n.$

25. Prove that ~~the~~ only ideals of a field F are {0} and F itself.

Let F be a field, then F is commutative with Unity
let S be an ideal of F such that
$S \neq \{0\}$

then: $\exists\, s \in S$ such that $s \neq 0$.

from this, we know,
$s^{-1} \cdot s = 1 \in S$.
$\therefore 1 \in S$.

then: ~~let~~ since S is ideal, it
absorb the multiple in F

then: let $r \in F$, then
$s \cdot r \in S$.
let $s = 1$,
$1 \cdot r = r \in S$. $\forall r \in F$

$\therefore S = F$.

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39. ~~[illegible]~~
Last problem set, we shown that
all subrings of $\mathbb{Z}$ are in the form $n\mathbb{Z} = \{n \cdot a \mid a \in \mathbb{Z}\}$
(Chapter 12, #13)

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and this is exactly all the subrings in $\mathbb{Z}$ is $n \cdot \mathbb{Z}$ where $n \in \mathbb{Z}$.

then: clearly this fits. the definition of $\langle a \rangle = \{ad \mid d \in \mathbb{Z}\}$

→ this is clearly ideal since ~~$n\mathbb{Z}$~~ $n \cdot a - n \cdot b = n(a-b) \in n\mathbb{Z}$.
and $r(n \cdot a)$ ~~a~~ $= n(r \cdot a) \in n\mathbb{Z}$.