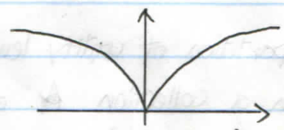
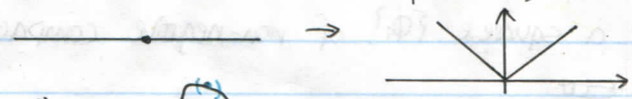
non-examples: 1. $\alpha: \mathbb{R} \to \mathbb{R}^2$ by $t \mapsto (t^3, t^2)$

$D\alpha = (3t^2, 2t)$ not rank 1 at $t=0$.



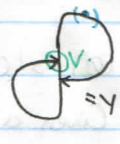
α is homo.

2. $t \mapsto (t^3, |t|^3)$.



3.

$\xrightarrow{\alpha}$ $D\alpha \neq 0$.



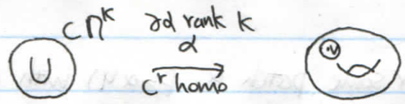
V contains vertical lines & $(x \to \infty)$.

not manifold.

∂M on friday.

Jan 20.

$\int_{[a,b]} f' = f|_b^a$.



$\subset \mathbb{R}^k$ $\partial\alpha$ rank k $\xrightarrow[C^r \text{ homo}]{\alpha}$

manifold with boundary.

Premitive. **Def.** Let $H^k = \{x \in \mathbb{R}^k, x_k \geq 0\}$ A k-manifold (of class C^r, possibly with boundary) is $M \subset \mathbb{R}^n$ s.t. each $p \in M$ has an open nbd (in M) s.t. there is an open $U \subset H^k$ & a C^r homemorphism $\alpha: U \to V$ whose differential is of rank k at every $x \in U$.

The ∂M ($\neq$ BdM.) of M is $\partial M = \{p \in M : \text{for some pitch } \alpha,\ p = \alpha(q) \text{ where } q \in \partial H^k\}$. ($\partial H^k = \mathbb{R}^{k-1} \times \{0\}$)

Claim: ∂M is also a manifold, without boundary, of dim $k-1$.

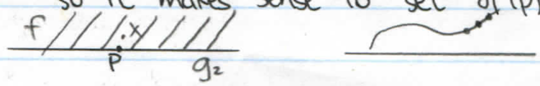
Issues: 1. What does differentiability mean on H^k?

2. Could it be the case that $\partial M = M$?

Def. Let $S \subset \mathbb{R}^k$ & $f: S \to \mathbb{R}^m$ we say that f is of class C^r on S if there exist an open $U \supset S$ and a C^r function $g: U \to \mathbb{R}$, s.t. $g|_S = f$ i.e. $\forall x \in S$, $f(x) = g(x)$.

Easy fact: If $f: H^k \to \mathbb{R}^m$ then $dF(p)$ is well-defined even for $p \in \partial H^k = \mathbb{R}^{k-1} \times \{0\}$ meaning, if g_1 and g_2 both extend ~~to~~ f to some open set containing H^k, then $(dg_1)(p) = (dg_2)(p)$. So it makes sense to set $\partial f(p) = \partial g_1(p)$.

proof:



Suprisingly hard fact: Diffability on a set S is a "local property": If $f: S \to \mathbb{R}$ has the property that for every $p \in S$ there is a nbd U_p & a diff. function $g_p: U_p \to \mathbb{R}$ s.t. $f|_{S \cap U_p} = g_p|_{S \cap U_p}$, then f is diffiable on S.

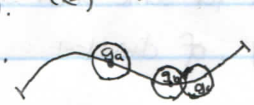
The partition of utility lemma: (Big theorm).
Given a collection $\mathcal{A}$ of open sets in $\mathbb{R}^n$ whose overall union is $A = \bigcup_{U \in \mathcal{A}} U$, there exist a sequence $\{\phi_i\}$ of non-negative compactly supported C^∞ functions s.t. $\text{supp}(\phi) = \{x : \phi(x) \neq 0\}$.

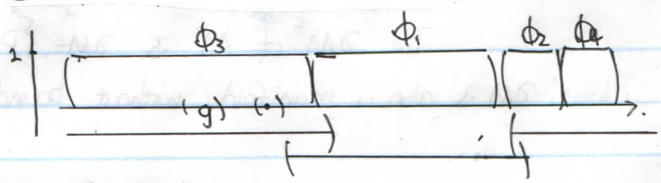
Compactly supported: $\text{supp}(\phi)$ is compact.
Partitions of Unity Lemma: Given a collection $\mathcal{A}$ of open sets in $\mathbb{R}^k$ whose overall union is $A = \bigcup_{U \in \mathcal{A}} U$. there exists a seq. $\{\phi_i\}$ of non-neg compactly-supp. C^∞ fcns s.t. 1. $\forall i.\ \exists U \in \mathcal{A}$ s.t. $\text{supp}(\phi_i) \subset U$.
2. Each $\text{supp}\,\phi_i$ is contained in one $U \in A$. 3. $\sum \phi_i(x) = 1$.

Jan 23.

$\partial M = \{\text{bndry pts}\} = \{p \in M : \text{for some patch } \alpha,\ p = \alpha(q) \text{ with } q \in \partial H^k := \mathbb{R}^{k-1} \times \{0\}\}$.
Thm. With M as before. ∂M is a $(k-1)$-manifold of class C^r with no boundary.
Def. $f: S \xrightarrow{\subset \mathbb{R}^k} \mathbb{R}$ is C^r differentiable if $\forall p \in S$, $\exists$ nbd U of p is $\mathbb{R}^k$ $\exists g: U \to \mathbb{R}$ s.t.
1. g is C^r. 2. $g(x) = f(x)$ for $x \in U \cap S$.

Thm.(2). Let S be an arbitrary subset of $\mathbb{R}^k$. A function $f: S \to \mathbb{R}^n$ is said to be of class C^r iff there exists an open sub.set U of $\mathbb{R}^k$ containing S and a C^r fcn $g: U \to \mathbb{R}^n$ such that $g(x) = f(x)$ for every $x \in S$.
proof: (of thm 2). ($\Leftarrow$) ✓
($\Rightarrow$) hard.





Partition of units: "division of labor".
Let $\mathcal{A} = \{U_{\text{open}} : F|_{S \cap U}$ has a diff able extension to U $\exists g: U \to \mathbb{R}^n$ 1. g is C^r. 2. $g(x) = f(x)$ for $x \in S \cap U\}$.
Then $A = \bigcup_{U \in \mathcal{A}} U$ is an open set containing S.
Find a POI. $\{\phi_i\}$ subordimte to $\mathcal{A}$. For each i find $U_i \in \mathcal{A}$ & a C^r. $g_i: U \to \mathbb{R}^n$ s.t.
$g_i|_{U_i \cap S} = f|_{U_i \cap S}$ wrong $g(x) = \sum_{i=1}^{\infty} \phi_i(x) \cdot g(x)$ extended by 0 to A.
g is diff able $g(x) = f(x)$ for $x \in S$. □