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Proof: Pick a basis $(z_i)_{i=1}^n$ of $N(T)$, extend to a basis $(z_i)_{i=1}^n \cup (v_j)_{j=1}^r$ of V and set $w_j = Tv_j$

Claim (w_j) is a basis of $R(T)$

Assumming claim

$\dim V = n + r = \text{nullity}(T) + \text{rank}(T)$

PF of claim: (w_j) spans.

Given $w \in R(T)$ find $v \in V$ s.t. $Tv = w$

as $(z_i) \cup (v_j)$ is a basis, I can find α_i & β_j s.t. $v = \sum \alpha_i z_i + \sum \beta_j v_j$ so

$$w = Tv = T(\sum \alpha_i z_i + \sum \beta_j v_j) = \sum \alpha_i \underbrace{Tz_i}_{0} + \sum \beta_j Tv_j$$

$$= \sum \beta_j w_j$$

(w_j) are l.i.:

Assume $\sum \beta_j w_j = 0$, so

$\sum \beta_j T v_j = 0$ so $T(\sum \beta_j v_j) = 0$

so $\sum \beta_j v_j \in N(T)$, so $\exists\, \alpha_i$ s.t.

$\sum \beta_j v_j = \sum \alpha_i z_i$

So $\sum \beta_j v_j + \sum(-\alpha_i z_i) = 0$.

but $(v_j) \cup (z_i)$ make a basis, so

$\forall i\ \alpha_i = 0$, $\forall j\ \beta_j = 0$.

In particular, in $\sum \beta_j w_j$ all coeffs are 0. So (w_j) are l.i. $\square$

Corollary If $\dim V = \dim W$, then TFAE (the following are equivalent)

1. T is 1-1 ($\Leftrightarrow N(T) = \{0\} \Leftrightarrow \text{nullity}(T) = 0$)
2. T is onto
3. $\text{rank}\, T = \dim V$
4. T is invertible.

Pf Let $n = \dim V = \dim W$, by thm

$$n = \operatorname{rank}(T) + \operatorname{nullity}(T)$$

$(1) \iff \text{nullity} = 0 \iff \text{rank} = n \ (3)$

$\iff \dim R(T) = \dim W$

$\iff R(T) = W \ (2)$

$\Updownarrow$

T is one to one & onto

$\iff T$ is invertible

claim If $T: V \to W$, $T': V' \to W'$
assume $\dim V = \dim V'$, $\dim W = \dim W'$
& also that $\operatorname{rank}(T) = \operatorname{rank}(T')$, Then
T & T' are isomorphic

Meaning $\quad T \underset{L}{\overset{R}{\rightleftarrows}} T'$