MAT1100 Homework 1

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Problem 1

Case 1: $|xy| = n < \infty$ for some $n \in \mathbb{N}$

Claim: $|yx| = n$.

Proof:

$$(yx)^{n+1} = y(xy)^n x = yx$$

Multiplying both sides by $(y)^{-1}$ on the left and x^{-1} on the right,

$$(yx)^n = e.$$

Since the order n of an element g in a group G is the smallest n such that $g^n = e$, we have $|yx| \leq n$ by the last equation. To show that $|yx| \geq n$, suppose $(yx)^k = e$ for some $k < n$ for a contradiction:

$$(xy)^{k+1} = x(yx)^k y = xy$$

Multiplying both sides by $(x)^{-1}$ on the left and y^{-1} on the right,

$$(xy)^k = e$$

.

This is a contradiction, since $|xy| = n > k$.

Therefore, we have proven the claim $|yx| = n$, and so $|xy| = |yx| = n$.

Case 2: $|xy| = \infty$.

If $|xy| = \infty$, then $(xy)^n \neq e$ for all $n \in \mathbb{N}$. Suppose that $|yx| = k < \infty$ for some $n \in \mathbb{N}$. Then

$$(xy)^{k+1} = x(yx)^k y = xy$$

Multiplying both sides by $(x)^{-1}$ on the left and y^{-1} on the right,

$$(xy)^k = e$$

.

This is a contradiction, since $|xy| = \infty$.

Therefore, $|xy| = |yx| = \infty$.

Problem 2

Let $x, y \in G$.
($\Rightarrow$) Suppose the function $\phi : G \to G$ given by $\phi(g) = g^2$ is a morphism of groups. Then

$$\phi(xy) = \phi(x)\phi(y) \Rightarrow (xy)^2 = x^2y^2$$

Multiplying both sides of the equation on the right by x^{-1} on the left and y^{-1} on the right,

$$yx = xy.$$

$x, y \in G$ was arbitrary, so G is abelian.

($\Leftarrow$) Suppose G is abelian. Then

$$\begin{aligned}\phi(xy) &= (xy)^2\\ &= xyxy\\ &= x(xy)y \text{ since } G \text{ is abelian}\\ &= \phi(x)\phi(y)\end{aligned}$$

Therefore, $\phi : G \to G$ given by $\phi(g) = g^2$ is a morphism of groups.

Problem 3

Let $G' = \{aba^{-1}b^{-1} | a, b \in G\}$, and we show that G' is a normal subgroup in G.

G' was defined as the subgroup generated by all the commutators of elements of G, so by definition, it is a subgroup of G. To show that G' is normal, let $x_1y_1{x_1}^{-1}{y_1}^{-1}...x_ny_nx_n^{-1}y_n^{-1} \in G'$ be arbitrary,with $x_1, y_1, ..., x_n, y_n \in G$, and let $a \in G$. Then

$$
\begin{aligned}
&a^{-1}x_1y_1{x_1}^{-1}{y_1}^{-1}...x_ny_nx_n^{-1}y_n^{-1}a \\
&= (a^{-1}x_1a)(a^{-1}y_1a)(a^{-1}{x_1}^{-1}a)(a^{-1}{y_1}^{-1}a)...a(a^{-1}x_na)(a^{-1}y_na)(a^{-1}x_n^{-1}a)(a^{-1}y_n^{-1}a) \\
&= (a^{-1}x_1a)(a^{-1}y_1a)(a^{-1}x_1a)^{-1}(a^{-1}y_1a)^{-1}...(a^{-1}x_na)(a^{-1}y_na)(a^{-1}x_na)^{-1}(a^{-1}y_na)^{-1} \in G' \\
&\text{since } (a^{-1}x_1a), (a^{-1}y_1a) \in G.
\end{aligned}
$$

Therefore, G' is normal.

To show that G/G' is abelian:
Let $x, y \in G$ be arbitrary. Then $x^{-1}y^{-1}xy = x^{-1}y^{-1}(x^{-1})^{-1}(y^{-1})^{-1}) \in G'$, so

$$x^{-1}y^{-1}xyG' = G'.$$

$$(yx)^{-1}xyG' = G'$$

Multiplying both sides by yx on the left,

$$xyG' = yxG'.$$

Therefore, G' is abelian.

Suppose ϕ is a morphism from G into an abelian group A, and we wish to show that there exists $\psi : G/G' \to A$ such that $\phi = \psi \circ \pi$, where $\pi : G \Rightarrow G/G'$ is the natural map $g \mapsto \bar{g} \in G/G'$. By the Universal Property of Quotients, it is sufficient to show that $G' \subset \ker \phi$.

Let $x_1y_1{x_1}^{-1}{y_1}^{-1}...x_ny_nx_n^{-1}y_n^{-1} \in G'$ be arbitrary. Then

$$
\begin{aligned}
\phi(x_1y_1{x_1}^{-1}{y_1}^{-1}...x_ny_nx_n^{-1}y_n^{-1}) &= \phi(x_1)\phi(y_1)\phi(x_1^{-1})\phi(y_1^{-1})...\phi(x_n)\phi(y_n)\phi(x_n^{-1})\phi(y_n^{-1}) \\
&= \phi(x_1)\phi(y_1)\phi(x_1)^{-1}\phi(y_1)^{-1}...\phi(x_n)\phi(y_n)\phi(x_n)^{-1}\phi(y_n)^{-1} \\
&= \phi(x_1)\phi(x_1)^{-1}\phi(y_1)\phi(y_1)^{-1}...\phi(x_n)\phi(x_n)^{-1}\phi(y_n)\phi(y_n)^{-1} \\
&\quad \text{since } \phi \text{ maps into A and A is abelian} \\
&= e
\end{aligned}
$$

Therefore, $xyx^{-1}y^{-1} \in \ker \phi$, so by the University Property of Quotients, there exists a $\psi : G/G' \to A$ such that $\phi = \psi \circ \pi$. i.e., any morphism from G into an abelian group factors through G/G'.

Problem 4

First, we show that Inn G is a subgroup of Aut G:
Let $\phi_g, \phi_h : x \mapsto x^g$ be inner automorphisms of G, and let $x \in G$. Then

$$
\begin{aligned}
\phi_g \circ \phi_{h^{-1}}(x) &= \phi_g(hxh^{-1}) \\
&= ghxh^{-1}g^{-1} \\
&= ghx(gh)^{-1} \\
&= \phi_{gh}
\end{aligned}
$$

is an inner automorphism of G.
Next, we show that Inn G is normal in Aut G:
Let $\phi_g : x \mapsto x^g$ be an inner automorphism and let $\psi \in$ Aut G. Then

$$
\begin{aligned}
\psi \circ \phi_g \circ \psi^{-1}(x) &= \psi \circ \phi_g(\psi^{-1}(x)) \\
&= \psi(g\psi^{-1}(x)g^{-1}) \\
&= \psi(g)\psi(\psi^{-1}(x))\psi(g^{-1}) \\
&= \psi(g)x\psi(g^{-1}) \\
&= \psi(g)x\psi(g)^{-1}
\end{aligned}
$$

is an inner automorphism of G, since $\psi(g) \in G$.
Therefore, Inn G is normal in Aut G.