November 17^{th}, 2009

MAT240: Abstract Linear Algebra Lecture:

Question 1: Can we choose P, Q so that PAQ is "simpler"?

Question 2: What's "simpler"?

Answer to Question 2:

Claim:

Let $B = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$ then $rank(B) = k$, where k is the number of row/columns of I

$T_B: F^n \to F^m, \quad v \to Bv$

$\boldsymbol{rank(B) = rank(T_B) = \dim(R(T_B))}$

$R(T_B) = span\{e_1 \dots e_k\}$

$rank(T_B) = k$

"Elementary Matrices"

Three kinds $E^1_{i,j}, E^2_{i,c}, E^3_{i,j,c}$

1. $E^1_{i,j}$ *= interchange row/column i with row/column j*
   $E^1_{i,j}A = A$ with rows i,j interchanged.
   $AE^1_{i,j} = A$ with columns i,j interchanged.
   Claim: $E^1_{i,j}$ is invertible and $\left(E^1_{i,j}\right)^{-1} = E^1_{i,j}$
   Proof: $E^1_{i,j}E^1_{i,j} = I$
2. $E^2_{i,c}$ *= multiply row or column I by a constant c (c cannot equal 0). If c = 0, then the matrix is not invertible.*
   $E^2_{i,c}A = A$ with row i multiplied by c.
   $AE^2_{i,c} = A$ with column i multiplied by c.
   Claim: $E^2_{i,c}$ is invertible and $\left(E^2_{i,c}\right)^{-1} = E^2_{i,c}$
   Proof: $E^2_{i,c}E^2_{i,c^{-1}} = I$

3. $E^3_{i,j,c} = \textit{add a multiple of one row to another}$.
   $E^3_{i,j,c}A = A$ with c times row j added to row i.

   $AE^3_{i,j,c} = A$ with c times column j added to column i.

   Claim: $\left(E^3_{i,j,c}\right)^{-1} = E^3_{i,j,-c}$

   Proof: (as above)

$$rank(A) = rank(\ldots E_4 E_1 A E_2 E_3 \ldots) = rank\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} = k$$

So:

$E^1_{i,j}A$: Interchange rows i &j

$E^2_{i,c}$A: Multiply row i by c

$E^3_{i,j,c}A$: Add c times row j to row i.

These are called "Elementary Row Operations"

Theorem: Every A can be r/c-reduced to $\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$:

E.g $A = \begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix} \xrightarrow{\frac{1}{4}R2, \frac{1}{8}R3, \frac{1}{6}R4} \begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & \frac{1}{4} & 0 & \frac{5}{4} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{4}{3} & \frac{1}{6} \end{pmatrix}$

$$\xrightarrow{R1 \leftrightarrow R2} \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 2 & 2 \\ 1 & \frac{1}{4} & 0 & \frac{5}{4} & \frac{1}{4} \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{4}{3} & \frac{1}{6} \end{pmatrix} \xrightarrow{R3-R1, R4-R1, \frac{1}{2}R2} \begin{pmatrix} 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & -\frac{3}{4} & -1 & -\frac{3}{4} & \frac{1}{4} \\ 0 & -\frac{1}{2} & -\frac{2}{3} & -\frac{2}{3} & \frac{1}{6} \end{pmatrix}$$

$$\xrightarrow{R1-R2, R3*-\frac{4}{3}, R4*-2} \begin{pmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{4}{3} & 1 & -\frac{1}{3} \\ 0 & 1 & \frac{4}{3} & \frac{4}{3} & -\frac{1}{3} \end{pmatrix} \xrightarrow{\dots} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Corollary 1:

$$rank(A) = rank(A^T)$$

Proof:

$$A \xrightarrow{r\backslash c \ operations} \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^T \xrightarrow{c\backslash r \ operations} \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}$$

Note: these are not the same matrix, just in the set format.

Comment: it is hard to interpret A^T in VS.

A^T as a linear transformation:

$$A \in M_{mxn} \rightarrow \begin{pmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{pmatrix} \rightarrow span(c_1, \dots c_n) = column - space(A)$$

$$A \in M_{mxn} \rightarrow \begin{pmatrix} \sim & r_1 & \sim \\ & \vdots & \\ \sim & r_n & \sim \end{pmatrix} \rightarrow span = row - space(A)$$

Corollary 2: $rank(A) = \dim(column - space(A)) = \dim(row - space(A))$

Proof 1 (deals with the first equals sign in the equation above):

$$\dim(R(T_A)) = rankA$$

$$T_A e_i = c_i$$

$$R(T_A) = span(c_1, \dots c_n)$$

$$\dim(\mathrm{R}(T_A)) = \dim(span(c_1, \dots c_n))$$

$$rank(A) = \dim(column - space(A))$$

Proof 2 (deals with the second equals sign in the equation above):

$$\dim(row - space(A)) = \dim(column - space(A^T)$$

$$rank(A^T) = rank(A)$$

$$= \dim(column - space(A))$$