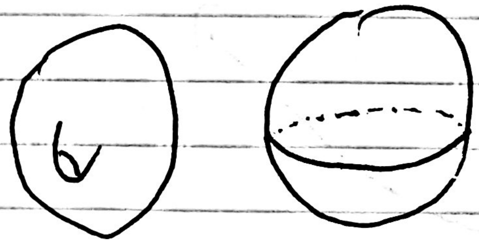
22th Wed March. Hour 066

Today: Integration. Stoke's theorem.

Read along: 34.35.37

If M is orinted, α and β on positive and $\operatorname{supp} W \subset \operatorname{im}\alpha, \operatorname{im}\beta$.
Then: $\int_{\mathbb{R}^k} \alpha^* w = \overset{\alpha}{\int_M} w = \overset{\beta}{\int_M} w = \int_{\mathbb{R}^k} \beta^* w = \int_M w$

$\int_M w$ in general: Chopped into pieces with measure-0, intersections/exceptions; parametrize and integrate then add together.
↓orinted



Example: Let S^2 be orinted as $\partial D^3 \supset \mathbb{R}^3$, and let $w \in \Omega^2(\mathbb{R}^3)$, given by $w \in x\,dy \wedge dz + (y\,dz \wedge dx + z\,dx \wedge dy$.

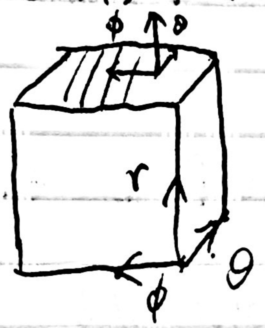
Compute $\int_{S^2} w$ (precisely: $\tau: S^2 \to \mathbb{R}^3$, $\tau^*: \Omega^2(\mathbb{R}^3) \to \Omega^2(S^2)$)

$\int_{S^2} \tau^* w \sim \int_{S^2} w$ use $\alpha: [0,\infty)_r \times [0, 2\pi]_\theta \times [-\frac{\pi}{2}, \frac{\pi}{2}]_\phi$
longitude latitute

$$\alpha(r,\theta,\phi) = (r\cos\theta\sin\phi,\ r\sin\theta\cos\phi,\ r\sin\phi)$$
$\det D\alpha = r^2\cos\phi > 0$, ~~so α is orientation~~
So α is orientation preserving positive.

Let $\beta = \alpha|_{r=1}$, $\beta(\theta,\phi) = (\cos\theta\cos\phi,\ \sin\theta\cos\phi,\ \sin\phi)$

$\alpha|_{[0,1]_r \times \cdots}$:



β also positive

β is positive, if $[0,2\pi]_\theta \times [-\frac{\pi}{2}, \frac{\pi}{2}]_\phi$ is taken with the order of orientation of (θ, ϕ)

$$\int_{S^2} w = \int_{\beta^*} w = \int_Q \underbrace{\cos\theta\cos\phi}_{x} (d\sin\theta\cos\phi) \wedge (d\sin\phi) + 2 \text{ other awful terms}$$

$$Q = [\ ]_\theta \times [\ ]_\phi \qquad = \int_Q \cos\phi \, d\theta \wedge d\phi$$

$$= \int_{[0,2\pi]_\theta \times [-\frac{\pi}{2}, \frac{\pi}{2}]_\phi} \cos\phi$$

$$= 2\pi \cdot 2 = 4\pi$$

In theory def of $\int_M w$ (M is ~~comp~~ compact oriented)

find a PO1 $\phi_i : M \to [0,1]$ smooth

Subordinate to positive charts of M.

* $\text{Supp}(\phi_i) \subset \text{image } \alpha$, α is a positive chart.

* $\Sigma \phi_i = 1$

* local finiteness

Def: $\int_M w = \sum_{i \in I} \int_M \phi_i w$ makes sense.

Prop: If ϕ_I and ψ_J are PO1, then ${}^{\phi}\!\int_M w = {}^{\psi}\!\int_M w$

Pf: ${}^{\phi}\!\int_M w = \sum_{i \in I} \int_M \phi_i w$ def $\qquad \doteq \int_M w$

$$= \sum_{i \in I} \int_M \Big(\sum_{j \in J} \psi_j\Big) \phi_i w \qquad \sum_{j=J} \psi_j = 1$$

$$= \sum_i \sum_j \int_M \psi_j \phi_i w$$

$$= \sum_j \sum_i \int_M \phi_i \psi_j w$$

$$= \sum_j \int_M \underbrace{\Big(\sum_i \phi_i\Big)}_{=1} \psi_j w$$

$$= \sum_j \int_M \psi_j w = {}^{\psi}\!\int_M w$$

Properties: $a_i \in \mathbb{R}$

1. $\int_M (a_1 w_1 + a_2 w_2) = a_1 \int_M w_1 + a_2 \int_M w_2$ $\square$

2. $\int_{-M} w = -\int_M w = \int_M -w$ $\square$

$-M$: M taken with opposite orientation

$$\boxed{\int_{\partial M} w = \int_M dw}$$