Apr 3

Preliminaries: 1. Poincaré's lemma: on $\mathbb{R}^n$ closed $\Rightarrow$ exact. If $d\omega = 0$ then $\exists \lambda$ s.t. $\omega = d\lambda$.

2. Integration by parts: if supp ω, η is compact $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg\omega} \int_{\mathbb{R}^n} d\omega \wedge \eta$.

$\Rightarrow (-1)^{\deg\omega} \int d(\omega \wedge \eta) = (-1)^{\deg\omega} \int d\omega \wedge \eta + \int \omega \wedge d\eta$.

$\int_M (\omega \wedge d\eta) = (-1)^{\deg\omega} \int_M d(\omega\wedge\eta) - (-1)^{\deg\omega} \int_M d\omega \wedge \eta = (-1)^{\deg\omega} \int_{\partial M} \omega \wedge \eta - (-1)^{\deg\omega} \int_M d\omega \wedge \eta$

if $\partial M = \emptyset$ or $=$ if $(\mathrm{supp}\,\omega) \cap \partial M = \emptyset$ $\quad - (-1)^{\deg\omega} \int_M (d\omega) \wedge \eta$.

$\dim(\Lambda^k(\mathbb{R}^n)) = \binom{n}{k} = \binom{n}{n-k} = \dim \Lambda^{n-k}(\mathbb{R}^n)$

$*: \Lambda^k \to \Lambda^{n-k}$ $\quad *(\phi_I) = \pm \phi_{I^c}$

In $\mathbb{R}^3$ $\quad *\phi_1 = \phi_{23}$ $\quad \phi_I \wedge *(\phi_I) = \phi_{\underline{n}}$

On $\mathbb{R}^4_{txyz}$ $\quad *(dt) = dx \wedge dy \wedge dz$ $\quad s=1$. $\quad dt \wedge s(dx \wedge dy \wedge dz) = dt\,dx\,dy\,dz$

$*(dx\,dy\,dz) = s\,dt = -dt$ $\quad dx \wedge dy \wedge dz \wedge (s\,dt) = dt\,dx\,dy\,dz$ $\quad s=-1$.

3. Hodge $*$: $\Omega^k(\mathbb{R}^n) \to \Omega^{n-k}(\mathbb{R}^n)$ s.t. If $\omega, \eta \in \Omega^k(\mathbb{R}^n)$ $\quad \langle \omega, \eta \rangle\, dx_{\underline{n}} = \omega \wedge (*\eta)$

$\omega = \sum f_I dx_I$ $\quad \langle\omega,\eta\rangle = \sum_I f_I \cdot g_I$ $\quad \eta = \sum g_I dx_I$.

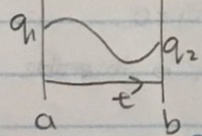
$\omega = dx_I$ $\quad \eta = dx_J$ $\quad LHS = \delta_{IJ}\, dx_{\underline{n}}$ $\quad RHS = \delta_{IJ}\, dx_{\underline{n}}$

Least Action principle: Every law in classical physics comes from minimizing some "action functional"

Maxwell's eqn. $\iff$ minimizing $S(A) = \int \frac{1}{2} \|dA\|^2 + J \wedge A$

E & M $\quad A \in \Omega^1(\mathbb{R}^4)$

Example. $C^\infty([a,b]) \ni q$. $q(a) = q_0$ $\quad q(b) = q_1$



trajectory of a particle.

Among all such fcns, find the extremum of $S(q) = \int_a^b \left(-\frac{1}{2} m \dot{q}^2 - V(q(t))\right) dt$.

kinetic energy $\quad$ potential energy

Extremes of $f: \mathbb{R}^n \to \mathbb{R}$ $\quad x$ is an extreme $\Rightarrow$ If you consider $f(x+\Delta x)$ where Δx is tiny then $\Delta f = f(x+\Delta x) - f(x)$ also tiny.

$\Delta f = f'(x)\Delta x + \frac{1}{2} f''(x) \Delta x^2 + \cdots$

x is an extremum of $f(x)$ $\Rightarrow$ $\varepsilon = 0$ is an extremum of $f(x + \varepsilon \Delta x)$

$\Rightarrow \frac{\partial f(x+\varepsilon\Delta x)}{\partial \varepsilon} = 0$ at $\varepsilon = 0$.

q is an extreme $\Rightarrow$ If $\delta q: [a,b] \to \mathbb{R}$ s.t. $\delta q(a) = \delta q(b) = 0$ then $\frac{\partial S(q + \varepsilon \delta q)}{\partial \varepsilon} = 0$ at $\varepsilon = 0$.

The Bare Necessities

1. `http://drorbn.net/1617-257` 2. This will be a tough class. 3. The essence: $\displaystyle\int_M d\omega = \int_{\partial M} \omega$, "Stokes' Theorem".

Like $\int_a^b f' = f|_a^b$, yet: What's M? What's ∂M? What's ω? What's $d\omega$? What's $\int$? Why true? Why care?

Preview: A Bit on Maxwell's Equations

Prerequisites.

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- Integration by parts: $\int \omega \wedge d\eta = -(-1)^{\deg \omega} \int (d\omega) \wedge \eta$ on domains that have no boundary.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q \mapsto \int_a^b \left(\frac{1}{2} m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

Table 18-1 Classical Physics

Maxwell's equations	
I. $\nabla \cdot E = \frac{\rho}{\epsilon_0}$	(Flux of E through a closed surface) = (Charge inside)/ϵ_0
II. $\nabla \times E = -\frac{\partial B}{\partial t}$	(Line integral of E around a loop) = $-\frac{d}{dt}$ (Flux of B through the loop)
III. $\nabla \cdot B = 0$	(Flux of B through a closed surface) = 0
IV. $c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$	c^2(Integral of B around a loop) = (Current through the loop)/ϵ_0 $+ \frac{\partial}{\partial t}$ (Flux of E through the loop)
[**Conservation of charge** $\nabla \cdot j = -\frac{\partial \rho}{\partial t}$	(Flux of current through a closed surface) = $-\frac{\partial}{\partial t}$ (Charge inside)]
Force law $F = q(E + v \times B)$	
Law of motion $\frac{d}{dt}(p) = F$, where $p = \frac{mv}{\sqrt{1 - v^2/c^2}}$	(Newton's law, with Einstein's modification)
Gravitation $F = -G \frac{m_1 m_2}{r^2} e_r$	

The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The *Vector Field* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} \|dA\|^2 \, dt dx dy dz + J \wedge A}$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d\star F = J}$$

These are the Maxwell equations! Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$dJ = 0 \Longrightarrow$	$\frac{\partial \rho}{\partial t} + \operatorname{div} j = 0$	"conservation of charge"
$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
	$\operatorname{curl} E = -\frac{\partial B}{\partial t}$	that's how generators work!
$d * F = J \Longrightarrow$	$\operatorname{div} E = -\rho$	"electrostatics"
	$\operatorname{curl} B = -\frac{\partial E}{\partial t} + j$	that's how electromagnets work!

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise. With $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e_2} (ds + eA)$ to derive Feynman's "law of motion" and "force law".