

Isometry which fixes 0 is orthogonal

When we did this in MAT247, I found it long and uninspiring.  
Now that I produced the proof on my own, it turned out to be trivial.

(THM) Let  $T$  be an isometry on  $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$  where  $\langle \cdot, \cdot \rangle$  is the usual inner product. If  $T(\vec{0}) = \vec{0}$ , then  $T$  is orthogonal, i.e.,  $T$  is a linear operator s.t.  $T^t T = I_n$ .

Pf: We divide it into two claims.

(Claim 1)  $T$  preserves inner product.

Pf of claim 1:  $T$  is isometry  $\Leftrightarrow \forall x, y \in \mathbb{R}^n \quad \langle T(x) - T(y), T(x) - T(y) \rangle = \langle x - y, x - y \rangle$ .<sup>(1)</sup>

Expanding both sides give  $\langle T(x), T(x) \rangle - 2\langle T(x), T(y) \rangle + \langle T(y), T(y) \rangle$   
 $= \langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle$ .<sup>(2)</sup>

Since  $T(\vec{0}) = \vec{0}$ . L.H.S of (2) =  $\langle x, x \rangle - 2\langle T(x), T(y) \rangle + \langle y, y \rangle$   
 $\Rightarrow \langle T(x), T(y) \rangle = \langle x, y \rangle \quad \forall x, y \in \mathbb{R}^n$  (In particular,  $T$  preserves norm of any vector)

Claim 2.  $T$  is linear

Pf of claim 2:  $\forall \lambda \in \mathbb{R}, x, y \in \mathbb{R}^n$

$$\begin{aligned} & \langle T(\lambda x + y) - \lambda T(x) - T(y), T(\lambda x + y) - \lambda T(x) - T(y) \rangle \\ &= \langle T(\lambda x + y), T(\lambda x + y) \rangle + \lambda^2 \langle T(x), T(x) \rangle + \langle T(y), T(y) \rangle \\ &\quad - 2\lambda \langle T(\lambda x + y), T(x) \rangle - 2\langle T(\lambda x + y), T(y) \rangle + 2\lambda \langle T(x), T(y) \rangle \\ &\stackrel{\text{Claim 1.}}{=} \langle \lambda x + y, \lambda x + y \rangle + \lambda^2 \langle x, x \rangle + \langle y, y \rangle - 2\lambda \langle \lambda x + y, x \rangle \\ &\quad - 2\langle \lambda x + y, y \rangle + 2\lambda \langle x, y \rangle \\ &= \lambda^2 \langle x, x \rangle + 2\lambda \langle x, y \rangle + \langle y, y \rangle + \lambda^2 \langle x, x \rangle + \langle y, y \rangle \\ &\quad - 2\lambda^2 \langle x, x \rangle - 2\lambda \langle y, x \rangle - 2\lambda \langle x, y \rangle - 2\langle y, y \rangle + 2\lambda \langle x, y \rangle \end{aligned}$$

$= 0$  by collecting like terms.

• by property of norm, <sup>we have</sup>  $T(\lambda x + y) - \lambda T(x) - T(y) = \vec{0}$ .  
 $\Leftrightarrow T(\lambda x + y) = \lambda T(x) + T(y) \quad \forall x, y \in \mathbb{R}^n$ .

From claim 1 and claim 2, we know. We can identify  $[T]_{\beta}$  with a matrix whose columns form an orthonormal set. ( $\beta$  here is the standard basis, with the usual order).

If you remember how to do inner product, and how to multiply matrices, it can be seen very easily.

$$[T]_{\beta}^t \cdot [T]_{\beta} = [I_n]_{\beta} \quad \square$$