Definition. $\mathcal{L}^k(V) = \{\underbrace{f: V^k \to \mathbb{R}}_{k\text{-linear forms}} : \forall 1 \le i \le k \quad \forall v_1, \dots, \hat{v}_i, \dots, v_k \in V$ we have that $v \mapsto f(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$ is linear$\}$

($\mathcal{L}^k$ is vector space)

Theorem. If $(a_1, \dots, a_n)$ is a basis of V, $\forall I \in \underline{n}^k = \{1, \dots, n\}^k$,

$\exists!\ \phi_I \in \mathcal{L}^k(V)$ such that $\phi_I(a_J) = \delta_{IJ}$, $\forall J \in \underline{n}^k$.

Furthermore, $\{\phi_I\}_{I \in \underline{n}^k}$ form a basis of $\mathcal{L}^k(V)$ and $\dim(\mathcal{L}^k(V)) = n^k$.

Definition. $\otimes: \mathcal{L}^k(V) \times \mathcal{L}^m(V) \to \mathcal{L}^{k+m}(V)$ is the tensor product.

If $f \in \mathcal{L}^k(V)$ and $g \in \mathcal{L}^m(V)$, $(f \otimes g)(v_1 \cdots v_{k+m}) = f(v_1 \dots v_k) \cdot g(v_{k+1} \cdots v_{k+m})$.

$\boxed{1}$ $\otimes$ is bilinear $\qquad$ $\boxed{2}$ $\otimes$ is associative.

$\boxed{3}$ $\phi_I(v_1 \cdots v_k) = \phi_{i_1}(v_1) \cdot \phi_{i_2}(v_2) \cdots \phi_{i_k}(v_k) = \phi_{i_1} \otimes \phi_{i_2} \otimes \dots \otimes \phi_{i_k}$.

Definition. Given a linear map $T: V \to W$, $T^*: \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ defn. by:

$T^* f(v_1 \dots v_k) = f(Tv_1 \dots Tv_k)$.

$\boxed{1}$ T^* is linear $\qquad$ $\boxed{2}$ T^* is contravariant $(S \circ T)^* = T^* \circ S^*$.

Definition. $\psi \in \mathcal{L}^k(V)$ is alternating $\Leftrightarrow \psi(\cdots x \cdots y \cdots) = -\psi(\cdots y \cdots x \cdots)$

$\Leftrightarrow \psi(\cdots x \cdots x \cdots) \equiv 0$

Definition. $S_n = \{\sigma: \underline{n} \to \underline{n} \mid \sigma$ is 1-1 and onto$\}$ permutation group of $\underline{n}$

Definition. $A^k(V) = \{\psi \in \mathcal{L}^k(V) : \psi$ is alternating$\}$

If $\psi \in \mathcal{L}^k(V)$, then $\psi \in A^k(V) \Leftrightarrow \forall \sigma \in S_k$, then we have:

$$\psi^\sigma(v_1 \dots v_k) = \psi(v_{\sigma 1} \cdots v_{\sigma k}) = (-1)^\sigma \cdot \psi(v_1 \cdots v_k).$$

$(-1)^\sigma$ is a unique sign assignment:

$\boxed{1}$ $(-1)^{ij} = (-1)$ $\qquad$ $(ij) \in S_n$ w/ $(ij)(k) = \begin{cases} j & k=i \\ i & k=j \\ k & o/w \end{cases}$

$\boxed{2}$ $(-1)^{\sigma \cdot \tau} = (-1)^\sigma \cdot (-1)^\tau$

$\boxed{3}$ $(-1)^\sigma = \det(A_\sigma)$, $A_\sigma = (e_{\sigma i})$.

Theorem. If $(a_1, \dots, a_n)$ is a basis of V, $\forall I \in \binom{n}{k}$,
$\exists! \, \Psi_I \in A^k(V)$ such that $\Psi_I(a_J) = \delta_{IJ}$, $\forall J \in \binom{n}{k}$.
Furthermore, $\{\Psi_I\}_{I \in \binom{n}{k}}$ form a basis of $A^k(V)$ and $\dim(A^k(V)) = \binom{n}{k}$.

Definition. $\Psi_I = \sum_{\sigma \in S_k} (-1)^\sigma \phi_I{}^\sigma \in \mathcal{L}^k(V). \Rightarrow \Psi_I \in A^k(V)$.

Claim. In $V = \mathbb{R}^n$, $a_i = e_i$, since $I \in \binom{n}{k}$, $\Psi_I(x_1 \dots x_k) = \det(X_I)$,
where $X_I = (x_1 | \dots | x_k)$.

Definition. $\wedge : A^k(V) \times A^l(V) \to A^{k+l}(V)$ is the wedge product.
If $f \in A^k(V)$ and $g \in A^l(V)$, then $f \wedge g \in A^{k+l}(V)$ is defn. by:

$$(f \wedge g)(x_1, \dots x_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma f(x_{\sigma 1} \cdots x_{\sigma k}) \cdot g(x_{\sigma(k+1)} \cdots x_{\sigma(k+l)})$$

$$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma 1 < \dots < \sigma k \\ \sigma(k+1) < \dots < \sigma(k+l)}} (-1)^\sigma f(x_{\sigma 1} \cdots x_{\sigma k}) \cdot g(x_{\sigma(k+1)} \cdots x_{\sigma(k+l)}).$$

$\boxed{1}$ $\wedge$ is bilinear $\boxed{2}$ $\wedge$ is associative

$\boxed{3}$ $\wedge$ is super-symmetric $f \in A^k$, $g \in A^l$ $f \wedge g = (-1)^{kl} g \wedge f$.

$\boxed{4}$ $\Psi_I = \phi_{i_1} \wedge \dots \wedge \phi_{i_k}$, where $I = (i_1, \dots, i_k)$.

Claim. $f \in A^k(W)$; $g \in A^l(W)$, then: $\underset{\in A^k(V)}{T^* f} \wedge \underset{\in A^l(V)}{T^* g} = \underset{\in A^{k+l}(V)}{T^*(f \wedge g)}$
$T: V \to W$; $T^*: A^k(W) \to A^k(V)$