October 8th

MAT 240 – Linear Algebra Lecture

If:

$G \subset V, \quad |G| = n, \quad span(G) = V$

L is linearly independent in V

Then:

$|L| \leq |G| \quad \& \quad \exists R \subset G \;\; s.t. \;\; |R| = |L| \quad \& \quad V = span\big((G\backslash R) \cup L\big)$

Proof of the above statement:

- By induction on $|L|$
- If $|L| = 0$ then $0 = |L| \leq |G|$ (as G is a natural number). With $R = \emptyset$ the conclusion holds/
- Assume lemma is known for any set L\` with $|L| = m + 1$

$$L = \{v_1 \quad \cdots \quad v_{m+1}\} \quad G = \{u_1 \quad \cdots \quad u_n\}$$

- Let $L\grave{} = \{v_1 \quad \cdots \quad v_m\}$, then L\` is linearly independent, $|L\grave{}| = m$, so by the induction hypothesis lemma holds for L\`. This implies $|L\grave{}| = m \leq |G| = n \quad \& \quad R\grave{} \subset G \; s.t. |R\grave{}| = |L\grave{}| \quad \& \quad span((G/R\grave{}) \cup L\grave{}) = V$
- Finally w.l.o.g. $R\grave{} = \{u_1 \quad \cdots \quad u_m\}$
- So now:

  $span(u_{m+1}, \dots u_n, v_1, \dots v_m) = V.$ In particular, $v_{m+1} \in span(\dots)$, that is, $\exists a_{m+1}, a_n$ and $b_1 \dots b_m \in F, s.t. \;\; V_{m+1} = \sum_{i=m+1}^{n} a_i u_i + \sum_{j=1}^{m} b_j v_j$. It cannot be that $\forall i, a_i = 0$, else we would have a linear dependency $V_{m+1} - \sum_{j=1}^{m} b_j v_m = 0$ among the elements of L, contradicting the assumption that L is independent. So that at least one $a_i = 0$ so $n \geq m + 1$, and also w.l.o.g $a_{m+1} \neq 0$ and so, by the preliminary statement; u_{m+1} is a linear combination of $u_{m+2}, \dots u_n$ and $v_1, \dots u_{m+1}$.

  Now take $R = \{u_{m+1}\} \cup R\grave{} = \{u_1 \dots u_{m+1}\}$

  Now $span(G\backslash R) \cup L) = span(\{u_{m+2}, \dots u_n, v_1, \dots v_{m+1}\}) \ni u_{m+1}$

  $= span(\{u_{m+1}, u_{m+2}, \dots u_n, v_1, \dots v_{m+1}\}) \supset span\{u_{m+1}, u_{m+2}, \dots u_n, v_1, \dots v_m\}$
  $= span(G\backslash R\grave{}) \cup L\grave{})$

$= V$

$\therefore span(G \backslash R`) \cup L`) = V$ ■

Corollaries:

1. If F has a finite basis β, then every other basis β_2 is also finite and $|\beta_1| = |\beta_2|$ (so dim(V) makes sense.

    Proof:

    Take $G = \beta_1$ assumeβ_2 is not finite & take L to be the first $|\beta_1| + 1$ elements of β_2, then L is linearly independent but $|L| \leq |G|$ contradicting Replacement Theorem.

    Now with $L = \beta_2$

    $\rightarrow |\beta_2| \leq |G| = |\beta_1|$

    $\rightarrow$ (Go through other half of proof again: $\beta_1 \leftrightarrow \beta_2$ ■

2. Assume $\dim(V) = n$
    a. If G generates V, $|G| \geq n$, if also $|G| = n$, then G is a basis.
    b. If $L \subset V$ is linearly independent, then $|L| \leq n$ & if also $|L| \leq n$ then L is a basis.