P16
21. Proof: $c \in F$

① For Z:

$+$: $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$

$\times$: $c(v_1, w_1) = (cv_1, cw_1)$

$\because$ V, W are vector spaces over the field F

$\therefore$ for V and W:

$+$: $V \times V \to V$ $\quad W \times W \to W$

$\times$: $F \times V \to V$ $\quad F \times W \to W$

that is:

$+$: $v_1 + v_2 = (v_1 + v_2)$ $\quad (v_1 + v_2) \in V$

$w_1 + w_2 = (w_1 + w_2)$ $\quad (w_1 + w_2) \in W$

$\times$: $c(v_1) = cv_1$ $\quad cv_1 \in V$

$c(w_1) = cw_1$ $\quad cw_1 \in W$

$\therefore$ $(v_1 + v_2, w_1 + w_2) \in Z$

$(cv_1, cw_1) \in Z$

$\therefore$ for Z:

$+$: $Z \times Z \to Z$

$\times$: $F \times Z \to Z$

Now we need to show Z follows VS1 – VS8.

② For VS1 (for all x, y in V, $x+y = y+x$)

For Z: $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ $\quad \begin{pmatrix} v_1, v_2 \in V \\ w_1, w_2 \in W \end{pmatrix}$

$(v_2, w_2) + (v_1, w_1) = (v_2 + v_1, w_2 + w_1)$

$\because$ V, W are vector spaces over F

$\therefore$ By VS1, In V and W

$v_1 + v_2 = v_2 + v_1$, $w_1 + w_2 = w_2 + w_1$

$\therefore$ $(v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1)$

$\therefore$ $(v_1, w_1) + (v_2, w_2) = (v_2, w_2) + (v_1, w_1)$

Hence Z follows VS1

③ For VS2 (For all x, y, z in V, $(x+y)+z = x+(y+z)$)

For Z: $[(v_1, w_1) + (v_2, w_2)] + (v_3, w_3)$ $\quad \begin{pmatrix} v_1, v_2, v_3 \in V \\ w_1, w_2, w_3 \in W \end{pmatrix}$

$= (v_1+v_2, w_1+w_2) + (v_3, w_3)$

$= (v_1+v_2+v_3, w_1+w_2+w_3)$

$(v_1, w_1) + [(v_2, w_2) + (v_3, w_3)]$

$= (v_1, w_1) + (v_2+v_3, w_2+w_3)$

$= (v_1+v_2+v_3, w_1+w_2+w_3)$

Hence $[(v_1, w_1) + (v_2+w_2)] + (v_3, w_3)$

$= (v_1, w_1) + [(v_2+w_2) + (v_3, w_3)]$

Z follows VS2.

④ For VS3 (There exists an element in V denoted by 0 s.t. $x+0 = x$ for each x in V)

$\because V, W$ are vector spaces over F

$\therefore$ By VS3: $x_v + 0_v = x_v$, $x_w + 0_w = x_w$

$x_v, 0_v \in V$, $x_w, 0_w \in W$

$\therefore (0_v, 0_w) \in Z$.

For Z: $(v_1, w_1) + (0_v, 0_w)$

$= (v_1 + 0_v, w_1 + 0_w)$

$= (v_1, w_1)$

Hence $\exists$ in Z, $0_z = (0_v, 0_w)$ s.t. $z + 0_z = z$ for each z in Z. Z follows VS3.

⑤ For VS4 (for each $x \in V$, $\exists\, y \in V$, s.t $x+y=$

$\because V, W$ are vector spaces over F

$\therefore$ By VS4, In V and W

for each $v_1 \in V$ and $w_1 \in W$

$\exists\, v_2 \in V$ and $w_2 \in W$

s.t. $v_1+v_2=0_V$ and $w_1+w_2=0_W$

For Z: for each (v_1, w_1) $\exists\, (v_2, w_2)$ s.t.

$(v_1+v_2, w_1+w_2) = (0_V, 0_W) = 0_Z$

Hence Z follows VS4

⑥ For VS5 (for each element x in V, $1 \cdot x = x$)

$\because V, W$ are vector spaces over F

$\therefore$ By VS5, In V and W

for each $v_1 \in V$ $w_1 \in W$

$1 \cdot v_1 = v_1$, $1 \cdot w_1 = w_1$

For Z: for each $(v_1, w_1) \in Z$.

$1 \cdot (v_1, w_1) = (1 \cdot v_1, 1 \cdot w_1) = (v_1, w_1)$.

Hence Z follows VS5.

⑦ For VS6 (For each pair of elements $a, b \in F$ and each $x \in V$, $(ab)x = a(bx)$.)

$\because V, W$ are vector spaces over F

$\therefore$ By VS6, In V, W