31th Mar. Friday Hour 070

Read along : ϕ Final material end at cpt 38.

$$\Omega^0 \longrightarrow \Omega^1 \longrightarrow \Omega^2 \xrightarrow{\text{Gauss' thm}} \Omega^3$$

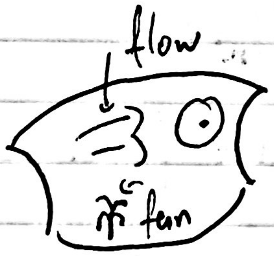
Fundemental theorem of the calculus, total elevation gain = $\int_{hike}$ climb rate

original Stokes' surface : S
vector field : a

degree to which vector circulation around ∂S of a.

$\int_S$ normal component of curl a

Pic :



Gauss : ~~Inp~~ Input ~~b~~ , b: v.f. D: a domain or solid in $\mathbb{R}^3$
c: function . S: surface

$W_2(b) = b_1 dx_2 \wedge dx_3 + C.P.$
$W_3(c) = c \cdot dx_1 \wedge dx_2 \wedge dx_3$
$dw_2(b) = W_3(\text{div} \cdot b)$

$\int_S w_2(b) = \int_S b \cdot \vec{n} \, dv$, $\int_D W_3(c) = \int_D c$

$W = w_2(b)$

$$\int_D dW_2(b) = \int_D dw = \int_{\partial D} w = \int_{\partial D} W_2(b) = \int_{\partial D} b \cdot \vec{n} \, dv$$

$$\shortparallel$$

$$\int_D W_3(\text{div} \cdot b) = \int_D \text{div}(b)$$

Gauss' thm / divergence thm

Let $w \in \Omega^k(M)$

1. w is "closed" if $dw = 0$ ∴ the integral of w on a boundary is 0. if $N^{k+1} \subset M$, $\int_{\partial N} w = \int_N dw = 0$

2. w is "exact" if $\exists \lambda \in \Omega^{k-1}(M)$ → if $N^k \subset M$, and $\partial N = \emptyset$ such that $w = d\lambda$, $\int_N w = \int_N d\lambda \underset{\text{stoke's}}{=} \int_{\partial N} \lambda = 0$

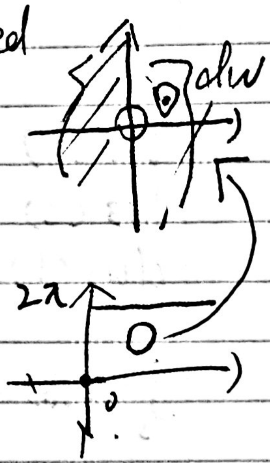
$$\Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1}$$

w is closed $\Longleftrightarrow$ $w = \ker d|_{\Omega^k}$
w is exact $\Longleftrightarrow$ $w \in \operatorname{im} d|_{\Omega^{k-1}}$

Comment (Claim): Every exact form is closed

Pf: w exact $\Rightarrow w = d\lambda$ for some $\lambda \Rightarrow dw = d(d\lambda) = 0$. ☐

Ex: $w = \dfrac{x\,dy - y\,dx}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$ w is closed



1. $dw = \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right) dx \wedge dy = 0$

$\rho(r,\theta) = (r\cos\theta, r\sin\theta)$

2. $\rho(dw) = d(\rho^* w) = d\left(\frac{r^2(\cos^2\theta + \sin^2\theta)\,d\theta}{r^2}\right)$
$= d(d\theta) = 0$

so dw is 0, because near every point $\in \mathbb{R}^2_{r,\theta}$, ρ is invertible

3. $w = d(\arctan \frac{y}{x}) = \arctan\left(\frac{-x}{y}\right) = \pm\frac{\pi}{2}$
$\Rightarrow dw = d(d\_\_) = 0$, except if $x = 0$

w is not exact ~ but nearly so, $w = d(\arctan y/x)\lambda$.
'nearly' but λ is not fully well-defined
indeed, $\int_{S^1} w = \int_{S^1} x\,dy - y\,dx = 2\pi \neq 0$, not exact.

Poincare's Lemma: On $\mathbb{R}^n$, every closed form is exact

De-Rharm: If M is compact, and closed
$\Rightarrow$ nearly exact.

$$H^k_{dR}(M) = \left\{ \{\text{closed } k\text{-forms}\} / \{\text{exact } k\text{-forms}\} \right\}$$ always finite dim.