Def $\mathcal{L}(V,W) := \begin{pmatrix}\text{the set of all} \\ \text{l.t. } V \to W\end{pmatrix} \subset \mathcal{F}(V,W)$

proposition $\mathcal{L}(V,W)$ is a vector space over $\mathbb{F}$ with

0. $\mathcal{L}(V,W) \ni 0_{\mathcal{L}(V,W)}$ defined by

$\forall u \in V \quad 0_{\mathcal{L}(V,W)}(u) = 0_W$

1. $T, S \in \mathcal{L}(V,W) \quad T+S : V \to W$

$u \in V \quad (T+S)(u) = T(u) +_W S(u)$

2. $(cT)(u) = c \cdot T(u)$

Pf (mostly skipped)

If T, S are l.t., is $T+S$ really a l.t.?

check (L is a l.t. $L(cu+v) = cL(u) + L(v)$)

$(T+S)(cu+v) \stackrel{?}{=} c(T+S)(u) + (T+S)(v)$

$LHS = T(cu+v) + S(cu+v) = cT(u) + T(v) + cS(u) + S(v)$

$RHS = c(T(u) + S(u)) + T(v) + S(v) = cT(u) + cS(u) + T(v) + S(v)$

$LHS = RHS$

claim The 'composition' of l.t is a l.t.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

"the composition of f & g"

$x \in X \quad (g\circ f)(x) = g(f(x))$.

If $U \xrightarrow{T} V \xrightarrow{S} W$

$U, V, W \,/\, F$, $T : U \to V$ & $S : V \to W$ are l.t

the $S\circ T : U \to W$ is a l.t.

$\underset{ST}{\shortparallel}$ "multiplication for l.t. not always defined"

Pf $(ST)(cx+y) = S(T(cx+y)) =$

$= S(cT(x)+T(y)) = c\,S(T(x)) + S(T(y))$

$= c(ST)(x) + (ST)(y)$

$\square$

Example

$U=V=W=P(\mathbb{R})$

$T = D : P(\mathbb{R}) \to P(\mathbb{R})$ $\quad Df = \frac{d}{dx}f = f'$

$S = \hat{x} : P(\mathbb{R}) \to P(\mathbb{R})$ $\quad \hat{x}f = x \cdot f$.

$TS \neq ST$. indeed. $(TS)(x^k) = T(x^{k+1}) = (k+1)x^k$

$(ST)(x^k) = S(kx^{k-1}) = kx^k$

Loosely A l.t. is fully determined by its values on a basis, and furthermore these values can be arbitrary

Precisely. Thm

If $V, W/\mathbb{F}$, if $\beta = (u_1 \cdots u_n)$ is a basis of V and if $w_1, \dots, w_n$ are any elements of W, then $\exists L : V \to W$ s.t. $L(u_i) = w_i \Rightarrow \mathcal{L}(V,W) \cong \{(w_1, \dots, w_n)\}$

Pf Given w_i set

$L(u) = \sum \alpha_i L(u_i) = \sum \alpha_i w_i$

whenever $u = \sum \alpha_i u_i$

Now check that L is indeed a l.t.

s.t. $L(u_i) = w_i$ $\square$