Algebraic Knot Theory: Homework #1 Solutions

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Problem 1

The Vassiliev knot invariants $\mathcal{V}_n$ are defined analogously to polynomials of degree n, in the sense that evaluation of an invariant on k-singular knots is similar to taking the k-th derivatives a polynomial. In fact, the analogy with polynomials runs even deeper than that. We show that if $f \in \mathcal{V}_n$ and $g \in \mathcal{V}_m$, then $f \cdot g \in \mathcal{V}_{m+n}$.

If $\mathcal{A}$ denotes the bialgebra of chord diagrams modulo the 4T relations, the induced weight system of the product is exactly what we would guess. That is, $W_{f \cdot g} = m_K \circ (W_f \otimes W_g) \circ \Box$, where K is the base field, m_K is multiplication in K, and $\Box$ is the coproduct on $\mathcal{A}$.

Solution

We need a combinatorial analog of the Leibniz rule, so that we can evaluate $f \cdot g$ on an N-singular knot. For this, write the singularities as $L_1 L_2 \cdots L_N$, and denote the right- and left-handed resolutions of the i-th crossing as L_i^+ and L_i^-, respectively. In the case of just one crossing, the relation $(f \cdot g)(L) = f(L^+)g(L) + f(L)g(L^-)$ is easy to see, just by expanding terms and canceling.

Given a (possibly empty) subset $\Lambda \subset \{1, \ldots, N\}$, define $\Lambda^+[L_1 \cdots L_N]$ to be the result of resolving L_i with L_i^+ for $i \in \Lambda$ and leaving the rest of the L_j alone. Define $\Lambda^-[L_1 \cdots L_N]$ to be the result of resolving L_i with L_i^- for $i \notin \Lambda$. We claim that the general formula is given by

$$(f \cdot g)(L_1 \cdots L_N) = \sum_{\Lambda \subset \{1,\ldots,N\}} f(\Lambda^+[L_1 \cdots L_N])g(\Lambda^-[L_1 \cdots L_N]).$$

We prove this by induction. The base case has been shown, so assume the above formula is true for N. Then, we may evaluate $f \cdot g$ on an $(N+1)$ singular knot as follows:

$$\begin{aligned}
&(f \cdot g)(L_1 \cdots L_{N+1}) = (f \cdot g)(L_1 \cdots L_N L_{N+1}^+) - (f \cdot g)(L_1 \cdots L_N L_{N+1}^-) \\
&= \sum_{\Lambda \subset \{1,\ldots,N\}} f(\Lambda^+[L_1 \cdots L_N]L_{N+1}^+)g(\Lambda^-[L_1 \cdots L_N]L_{N+1}^+) - f(\Lambda^+[L_1 \cdots L_N]L_{N+1}^-)g(\Lambda^-[L_1 \cdots L_N]L_{N+1}^-) \\
&= \sum_{\Lambda \subset \{1,\ldots,N\}} f(\Lambda^+[L_1 \cdots L_N]L_{N+1}^+)g(\Lambda^-[L_1 \cdots L_N]L_{N+1}) + f(\Lambda^+[L_1 \cdots L_N]L_{N+1})g(\Lambda^-[L_1 \cdots L_N]L_{N+1}^-) \\
&= \sum_{\Lambda \subset \{1,\ldots,N+1\}} f(\Lambda^+[L_1 \cdots L_N L_{N+1}])g(\Lambda^-[L_1 \cdots L_N L_{N+1}]).
\end{aligned}$$

This completes the induction. Now, suppose we evaluate $f \cdot g$ on a knot with more than $m+n$ singularities. Each term in the expansion we just derived must have more than n singularities evaluated in f or more than m singularities evaluated in g, so each term will vanish. Therefore, $f \cdot g \in \mathcal{V}_{m+n}$.

Next, consider the weight system $W_{f \cdot g}$, evaluated on a chord diagram D of degree $m+n$. By the formula we derived above, all terms must vanish except the terms with n singularities in f and m singularities in g. That is, $W_{f \cdot g}(D)$ is the sum of all terms in our formula where $|\Lambda| = m$. This is obtained by dividing the chords of D into sub-diagrams of degrees n and m, evaluating the former with W_f and the latter with W_g, then summing the product of these over all such divisions.

However, tracing $(m_K \circ (W_f \otimes W_g) \circ \Box)(D)$ through each step, we get the same result. Explicitly, $\Box(D)$ is the sum of all the splittings of D (by a splitting, we mean a division of the chords of D into two tensored sub-diagrams $D_1 \otimes D_2$). Then, $m_K \circ (W_f \otimes W_g)$ will vanish on each term except those with splittings into groups of n and m, over which it sums to the value of $W_{f \cdot g}(D)$. Hence, $W_{f \cdot g} = m_K \circ (W_f \otimes W_g) \circ \Box$.

Problem 2

By the Milnor-Moore theorem, we know that the bialgebra $\mathcal{A}$ is isomorphic to the polynomial bialgebra over its primitive elements. Modding out by the theta graph (a primitive element), we can construct a quotient map $\mathcal{A} \to \mathcal{A}^r := \mathcal{A}/\langle\theta\rangle$. Under the Milnor-Moore isomorphism, $\mathcal{A}^r$ includes into $\mathcal{A}$ as those polynomials with no θ terms. The composition of these two maps is a map $p : \mathcal{A} \to \mathcal{A}$, which can be viewed as evaluating $\theta = 0$, under this isomorphism. However, we don't know how to write down the isomorphism, so we try to construct p by different means.

Consider the element $W_\theta \in \mathcal{A}^*$, dual to θ. That is $W_\theta(\theta) = 1$, and W_θ maps diagrams of degree > 1 to 0. Then, let $W_\theta^* : \mathcal{A} \to A$ be the map adjoint to multiplication by W_θ on $\mathcal{A}^*$. This process mimics the construction of the momentum operator in quantum mechanics, so we suggestively use the notation $\partial_\theta = W_\theta^*$. Our best guess for the map p would be a "Taylor expansion of a polynomial about θ, evaluated at $\Delta\theta = -\theta$." So, let us define $P : \mathcal{A} \to \mathcal{A}$ as

$$P = \sum_{n=0}^{\infty} \frac{(-\theta)^n}{n!} \partial_\theta^n.$$

We check that ∂_θ behaves like a derivative, and we verify that P satisfies some of the properties that we would expect for p.

1. The so-called canonical commutation relation holds: $[\partial_\theta, \theta] = \mathbb{1}$.

2. P is a degree 0 operator: $\deg P(a) = \deg a$ for all $a \in \mathcal{A}$.

3. ∂_θ satisfies Lebniz' rule: $\partial_\theta(ab) = (\partial_\theta a)b + a(\partial_\theta b)$ for all $a, b \in \mathcal{A}$.

4. P is an algebra morphism: $P(1) = 1$ and $P(ab) = P(a)P(b)$.

5. θ satisfies the co-Lebniz rule: $\Box \circ \theta = (\theta \otimes \mathbb{1} + \mathbb{1} \otimes \theta) \circ \Box$.

6. P is a co-algebra morphism: $\eta \circ P = \eta$ (where η is the co-unit of $\mathcal{A}$) and $\Box \circ P = (P \otimes P) \circ \Box$.

7. $P(\theta) = 0$, hence $P(\langle\theta\rangle) = 0$.

8. If $Q : \mathcal{A} \to \mathcal{A}$ is defined by

$$Q = \sum_{n=0}^{\infty} \frac{(-\theta)^n}{(n+1)!} \partial_\theta^{n+1},$$

then $a = \theta Q(a) + P(a)$ for all $a \in \mathcal{A}$.

9. $\ker P = \langle \theta \rangle$.

10. P descends to a Hopf algebra morphism $\mathcal{A}^r \to \mathcal{A}$, and if $\pi : \mathcal{A} \to \mathcal{A}^r$ is the obvious projection, then $\pi \circ P$ is the identity of $\mathcal{A}^r$.

11. $P^2 = P$.

Solution

First, we give a combinatorial interpretation of the adjoint map $\partial_\theta = (W_\theta \otimes \mathbb{1}) \circ \square$. Since W_θ kills any chord diagram except θ, ∂_θ maps a chord diagram to the sum of each way of removing one chord from the diagram.

From this, we see that $\partial_\theta(ab) = (\partial_\theta a)b + a(\partial_\theta b)$, since this just corresponds to breaking up the sum into terms where we kill a chord coming from a or b. This gives us (3). It is clear that $\partial_\theta \theta = 1$, so $\partial_\theta(\theta D) = D + \theta \partial_\theta D$, which gives us (1) after rearranging terms.

Next, note that $P(D)$ is always a finite sum, since D is killed by all derivatives of higher order than its degree. Hence, P is well-defined (similarly, Q is well-defined). Also, ∂_θ decrements the degree of a diagram by 1, but multiplication by θ increases degree by 1. So, every term of P preserves degree, giving us (2). Also, every term of P except the constant term kills 1 (i.e. the empty diagram), so $P(1) = 1$.

Since ∂_θ satisfies Leibniz' rule and $\mathcal{A}$ is commutative, it can easily be shown inductively that

$$\partial_\theta^n(ab) = \sum_{i=0}^{n} \binom{n}{i} \partial_\theta^i a \partial_\theta^{n-i} b.$$

However, by Cauchy multiplication, we see that the coefficient of $(-\theta)^n$ in $P(a)P(b)$ is just

$$\sum_{i=0}^{n} \frac{\partial_\theta^i a}{i!} \frac{\partial_\theta^{n-i} b}{(n-i)!} = \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} \partial_\theta^i a \partial_\theta^{n-i} b,$$

so $P(ab) = P(a)P(b)$, with equality in each coefficient of the expansions. This establishes (4).

Property (5) is seen as follows. The right-hand side of the equation corresponds to taking the sum of two copies of all splittings of a diagram, then adding one chord to the left multiplicand of one copy and one cord to the right multiplicand of the other copy of each splitting. The left-hand side of (5) is given by adding one chord to the diagram first, then summing over all splittings, which yields the same result (since the extra chord will end up either on the left or on the right multiplicand when we split). We call this the "co-Leibniz" rule since it is has the same form as the Leibniz rule ($\partial_\theta \circ \nabla = \nabla \circ (\partial_\theta \otimes \mathbb{1} + \mathbb{1} \otimes \partial_\theta)$), with all the arrows reversed.

Since η just picks off the degree 0 part of an element of $\mathcal{A}$, P preserves degree, and P acts as the identity on degree 0 elements, we have $\eta \circ P = \eta$. Next, note that $P(D)$ has the combinatorial interpretation of the sum (over n) of diagrams obtained by removing n chords from D and adding them back in as isolated chords. So, $(\square \circ P)(D)$ is the sum of all possible splittings of diagrams obtained by replacing a group of chords in

D with an isolated group of chords. Next, $((P \otimes P) \circ \square)(D)$ is the sum over all possible ways to split the chords in D, further summed over all possible ways to replace groups of chords in each tensor multiplicand by an isolated group of chords. Both operators yield the same sum, so we have (6).

It is easy to see that $(-\theta)Q$ is just P without the $i = 0$ term, so $P = \mathbb{1} - \theta Q$, which gives us (8).

$P(\theta) = 1 - \theta \partial_\theta \theta = 0$, since all higher derivatives kill θ. Therefore, $\langle \theta \rangle \subset \ker P$. Conversely, if $a \in \ker P$, (8) gives us $a = \theta Q(a) \in \langle \theta \rangle$. This establishes (7) and (9).

Property (9) immediately implies that P induces a morphism $\mathcal{A}^r \to \mathcal{A}$, since $\mathcal{A}^r = \mathcal{A}/\langle \theta \rangle$. Furthermore, (8) implies that $P(a) \equiv a \pmod{\theta}$, so $\pi \circ P$ is the identity on $\mathcal{A}^r$, giving us (10).

Finally, (8) implies that $P(P - \mathbb{1})(a) = P(-\theta Q(a)) = 0$, which establishes (11).