Problem 5. (Klein's 1983 course) Show that $\mathbb{Z}[\sqrt{10}]$ is not a UFD.

Consider the element $9 \in \mathbb{Z}[\sqrt{10}]$. $9 = 3 \cdot 3$ is a factorization of 9 and 3 is an prime number. If $\mathbb{Z}[\sqrt{10}]$ is a UFD then it must also be an Integral Domain. In any Integral Domain all prime elements are irreducible and since 3 in $\mathbb{Z}[\sqrt{10}]$ is prime it is also irreducible in $\mathbb{Z}[\sqrt{10}]$.

Notice that $9 = (7 + 2\sqrt{10})(7 - 2\sqrt{10})$ is also a factorization of 9. The norm of any $a + b\sqrt{10} \in \mathbb{Z}[\sqrt{10}]$ is given by $N(a + b\sqrt{10}) = a^2 - 10b^2$. Since the norm of $(7 + 2\sqrt{10})$ is $7^2 - 10 \cdot 2^2 = 9$, if there exist some $x, y \in \mathbb{Z}[\sqrt{10}]$ such that $(7+2\sqrt{10}) = xy$ then the norm of x must equal to 3 meaning that there must be some integers a, b such that $a^2 - 10b^2 = 3$. No such integers exist because $a^2 - 10b^2 = 3$ means that $a^2 \equiv 3$ $(mod\ 10)$ and no such number exists ($0^2 \equiv 0,\ 1^2 \equiv 1,\ 2^2 \equiv 4,\ 5^2 \equiv 5, 6^2 \equiv 6,\ 7^2 \equiv 9,\ 8^2 \equiv 4,\ 9^2 \equiv 1$ all $(mod 10)$). Therefore $(7+2\sqrt{10})$ is irreducible in $\mathbb{Z}[\sqrt{10}]$ and so is $(7-2\sqrt{10})$ because it also has a norm of 9.

Therefore 9 can be factored into irreducibles in two distinct ways and since $9 \in \mathbb{Z}[\sqrt{10}]$, $\mathbb{Z}[\sqrt{10}]$ is not a UFD.$\square$