7th, Mon March Hour 059 Read 29-31

Thm: $\exists !$ linear $d: \Omega^k \to \Omega^{k+1}$ such that

1. $F \in \Omega^0 \Rightarrow (dF)(\xi) = D_\xi F$
2. $d(w \wedge \eta) = \sum \frac{\partial F}{\partial x_i} dx_i = (dw) \wedge \eta + (-1)^{\deg(w)} w \wedge (d\eta)$
3. $d^2 = 0$

Pf: 1-3 imply $d(\sum a_I dx_I) = \sum_{j,I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I$

$$\begin{array}{ccccccc}
\Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \to & \Omega^2(\mathbb{R}^3) & \to & \Omega^3(\mathbb{R}^3) \\
\{F\} & & \{a_1 dx_1 + a_2 dx_2 + a_3 dx_3\} & & \{b_1 dx_2 \wedge dx_3 + \cdots\} & & \{dx_1 \wedge dx_2 \wedge dx_3\} \\
\updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
\{F\} & \xrightarrow[F \mapsto \text{grad} F]{\text{div of max ascent}} & \left\{\begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix}\right\} & \xrightarrow[A \mapsto \text{curl}(A)]{} & \left\{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\right\} & \xrightarrow[\nabla \cdot B]{\text{div}\cdot B} & \{c\} \\
& \nabla & & \nabla \times A & & &
\end{array}$$

$$F \longmapsto \begin{Bmatrix} \frac{\partial F}{\partial x_1}, \\ \frac{\partial F}{\partial x_2}, \\ \frac{\partial F}{\partial x_3} \end{Bmatrix} \qquad \longmapsto \begin{Bmatrix} \frac{\partial a_2}{\partial x_3} - \frac{\partial a_1}{\partial x_2} \\ \vdots \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \longmapsto \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3}$$

$0 \mapsto 1$: If $F: \mathbb{R}^n \to \mathbb{R}$,
grad F = direction in which F increases the ~~fas~~ fastest.

Pf: $D_{(x,v)} F = \sum \frac{\partial F}{\partial x_i}(x) \cdot v_i = \langle \text{grad} F, v \rangle$
$\quad = \sum (\frac{\partial F}{\partial x_i} dx_i)(v)$ //

where grad F maximal if v points in their direction of grad F.

$1 \mapsto 2$: $d(a_1 dx_1 + a_2 dx_2 + a_3 dx_3)$

$= \frac{\partial a_1}{\partial x_1} \cancel{dx_1 \wedge dx_1} + \frac{\partial a_1}{\partial x_2} dx_2 \wedge dx_1 + \frac{\partial a_1}{\partial x_3} dx_3 \wedge dx_1 + \dots$ 6 more

$= -\frac{\partial a_1}{\partial x_2} dx_1 \wedge dx_2 + \frac{\partial a_1}{\partial x_3} dx_3 \wedge dx_1 + $ 4 more

$$\begin{pmatrix} \frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1} \\ \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \end{pmatrix} \begin{matrix} dx_2 \wedge dx_3 \\ dx_3 \wedge dx_1 \\ dx_1 \wedge dx_2 \end{matrix}$$

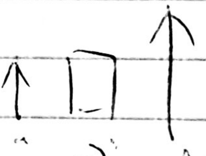
curl measures rotation of a body immersed in a flow of water

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

part of rotation around z.

Ex: 

in $\mathbb{R}^2$, rotate?

1. flow left < flow right  $\frac{\partial a_2}{\partial x_1}$ grows in x direction

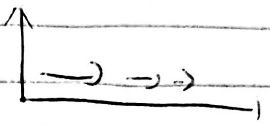
2.  $-\frac{\partial a_1}{\partial x_2}$ in y direction

$2 \mapsto 3$: creation / annihilation of flow

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \mapsto \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3}$$

for real water in $\mathbb{R}^3$

$\mathrm{div}(B) = 0$

$\frac{\partial b_1}{\partial x_1} < 0$: 

B: 

$\frac{\partial b_1}{\partial x_1} > 0$: 

measures water added to the system

$d^2 = 0$: curl(−grad F) = 0 div. (curl (A)) = 0

Claim: what remains for pf of thm
If for $w = \sum a_I dx_I \in \Omega^k(\mathbb{R}^n)$,
we set $dw = \sum_{i, I} \frac{\partial a_I}{\partial x_j} dx_j \wedge dx_I$

then d satisfies 1-3

a_I: 0-form, dx_j: 1-form dx_I: k-form

$$\sum_{j\, I} dx_j \wedge \frac{\partial a_I}{\partial x_j} \cdot dx_I$$

$$= \sum_j dx_j \wedge \frac{\partial w}{\partial x_j} = dw.$$