Tutorial
26.10.06

$P_n(\mathbb{R}) = \{\text{pol of deg} \le n\}$

$W = \{f \in P_n(\mathbb{R}) \mid f(a) = 0\}$

[we understand now that this W is the kernel of linear transformation $T: f \longmapsto f(a)$]

Let us produce a basis

$S = \{(x-a), (x-a)^2, \ldots (x-a)^n\}$ (guessing)

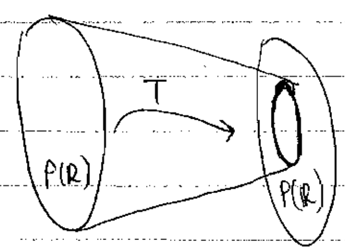
there n vectors in $P_n(\mathbb{R})$ and they are linearly indep.
$\downarrow$
n+1 dimentional

if S is not a basis for W, then $\exists$ $v \in W$ s.t. v is lin. indep. on vectors in S. But then $\dim W = n+1 \Rightarrow W = P(\mathbb{R})$ ↯

*) $T: P(\mathbb{R}) \to P(\mathbb{R})$

$$T(f(x)) = \int_0^x f(t)\,dt$$

Prove that T is one-to-one but not onto



Proof: ① One-to-one means $\ker T = \{0\}$

② Not onto means $R(T) \neq P(\mathbb{R})$

1) $\ker T = \{ f \in P(\mathbb{R}) \mid T(f) = 0 \}$

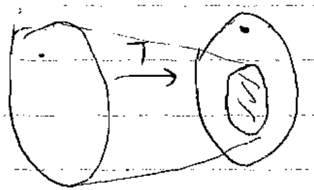
$= \{ f \in P(\mathbb{R}) \mid \int_0^x f(t)\,dt = 0 \} \overset{①}{=}$

Note that if f has degree n, then $\int_0^x f(t)\,dt$ is again a polynomial of degree $(n+1)$ $\quad \int_0^x a_i t^i\,dt = b_i t^{i+1} \Big|_0^x = b_i x^{i+1}$ $\quad (\forall f \in P(\mathbb{R}) \ \deg T(f) \geq 1$

$\overset{②}{=} \{ f \in P(\mathbb{R}) \mid f = 0 \} = \{0\} = \ker T$

2) We have to prove that $\exists\ f \in P(\mathbb{R})$ s.t. $\nexists\ g \in P(\mathbb{R})$ with

$T(g) = f$



Claim: constants $\mathbb{R} \subset P(\mathbb{R})$ are not in the range.

$\deg T(\mathbb{R}) \geq 1$