MAT257 Midterm 1 Review

COMPACTNESS, CONNECTEDNESS

COVERING

Definition. Let X be a subspace of $\mathbb{R}^n$. A covering of X is a collection of subsets of $\mathbb{R}^n$ whose union contains X; if each of the subsets is open in $\mathbb{R}^n$, it is called an open covering of X.

COMPACT

Definition. The space X is said to be compact if every open covering of X contains a finite subcollection that also forms an open covering of X, i.e.:

A metric space X is called compact if whenever you cover X with open sets, finitely many of them already cover X.

$$\left[X = \bigcup_{\alpha \in I} U_\alpha,\ U_\alpha \text{ open}\right] \Rightarrow \left[\exists F \subseteq I \text{ finite, s.t. } X = \bigcup_{\alpha \in F} U_\alpha\right]$$

THM. A continuous function on a compact space is bounded, meaning $\exists M$ s.t. $\forall x \in X$ $|f(x)| < M$.

Definition. A subset $A \subseteq X$ is a compact subspace if whenever you cover A with open sets, open in X or open in A, you can find a finite subcover, meaning a finite number of those open sets already cover A.

In other words, $A \subseteq X$ compact means:

$$\left(A = \bigcup_{\alpha \in I} U_\alpha,\ U_\alpha \text{ open in } A\right) \Rightarrow \left(\exists F \subseteq I \text{ finite, such that } A = \bigcup_{\alpha \in F} U_\alpha\right)$$

$$\Updownarrow$$

$$\left(A \subseteq \bigcup_{\alpha \in I} V_\alpha,\ V_\alpha \text{ open in } X\right) \Rightarrow \left(\exists F \subseteq I \text{ finite, such that } A \subseteq \bigcup_{\alpha \in F} V_\alpha\right)$$

Ex: Finite, $[0,1]$, $[a,b]$.

CounterEx: Uncountably infinite, $(0,1)$.

Theorem. $[0,1]$ is compact.

Proof. Let $\mathcal{U}_\alpha$ be open set such that $\bigcup \mathcal{U}_\alpha = I = [0,1]$.

Let $G = \{y \in [0,1] :$ the interval from 0 to y can be covered by finitely many $\mathcal{U}_\alpha$'s. $\}$

Then $0 \in G$, since $[0,0] = \{0\}$ is a singleton w/ $\{0\} \subset I$.

So, $\exists \alpha_0$ such that $0 \in \mathcal{U}_{\alpha_0}$ and then $[0,0] \subset \mathcal{U}_{\alpha_0}$, so $0 \le g \le 1$.

$[0,0]$ has a finite subcover. G is bd.d & nonempty, so $g = \sup G$ exists.

Claim. $g \in G$.

Proof. Since $g \in \bigcup \mathcal{U}_\alpha$, we can find some α_0 such that $g \in \mathcal{U}_{\alpha_0}$.

$\mathcal{U}_{\alpha_0}$ is open, so $\exists \epsilon > 0$ such that $(g-\epsilon, g+\epsilon) \subset \mathcal{U}_{\alpha_0}$.

Since $g = \sup G$, $\exists y \in G$ such that $g - \epsilon < y \le g$, but then by definition of G, $\exists$ finitely many $\mathcal{U}_{\alpha_1}, \dots, \mathcal{U}_{\alpha_n}$ such that $[0,y] \subseteq \bigcup_{i=1}^{n} \mathcal{U}_{\alpha_i}$.

Then, $\bigcup_{i=0}^{n} \mathcal{U}_{\alpha_i} \supseteq (g-\epsilon, g+\epsilon) \cup [0,y] \supseteq [0,g]$.

So, $[0,g]$ has a finite cover by $\mathcal{U}_\alpha$'s so $g \in G$.

Claim. $g > 0$

Proof. Some $\mathcal{U}_{\alpha_0}$ covers 0. It also covers $[0,\epsilon)$ for some $\epsilon > 0$.

$\Rightarrow \sup G \geq \epsilon > 0 \Rightarrow g > 0$.

Claim. $g = 1$

Proof. If not, $g < 1$. Then, some $\mathcal{U}_{\alpha_0}$ covers g, some $\mathcal{U}_{\alpha_1}, \dots, \mathcal{U}_{\alpha_n}$ cover $[0,g]$, since $\mathcal{U}_{\alpha_0}$ is open, it covers $(g-\epsilon, g+\epsilon)$ for some $\epsilon > 0$.

$\bigcup_{i=1}^{n} \mathcal{U}_{\alpha_i} \supseteq [0, g+\epsilon) \supseteq [0, g+\frac{\epsilon}{2}]$, but $g + \frac{\epsilon}{2} \in G$ and $g + \frac{\epsilon}{2} > g$, which is a contradiction since $\sup G = g$.

Thus, $g = 1$. $\square$

Definition. X is bounded $\Leftrightarrow \exists M. \forall x \in X. \|x\| < M \Leftrightarrow \exists N. \forall x \in X. |x| < N$.

THM. $X \subseteq \mathbb{R}^n$ is compact IFF X is closed and bounded.

Proof. ($\Rightarrow$) If $X \subseteq \mathbb{R}^n$ is compact, then: $X \subset \bigcup_{k=1}^{\infty} U(0;k) = \mathbb{R}^n$

$\Rightarrow X \subseteq \bigcup_{k=1}^{N} U(0;k) = U(0,N)$, so bdd. ($\|x\|$ cont $\Rightarrow$ bdd on X).

WTS X closed and X^c open.

Let $x \notin X$. For any k, consider $D_k = \{y : d(x,y) > \frac{1}{k}\}$.

Show D_k is open: $\bigcup_{k=1}^{\infty} D_k = \mathbb{R}^n \setminus \{x\} \supseteq X$.

By compactness, $\exists N$ such that $\bigcup_{k=1}^{N} D_k \supseteq X$.

NOT DONE

But, $\bigcup_{k=1}^{N} D_k = D_N$. So, $X \subseteq D_N$ and $\mathbb{R}^n \setminus \{x\} \supseteq \mathbb{R}^n \setminus D_N \supset U(x, \frac{1}{N})$

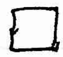
So, X^c open and X closed.

($\Leftarrow$) Harder ... boxes are compact.

Definition. Suppose (X, d_1) and (Y, d_2) are metric spaces. Define $X \times Y$ =

$d((x,y),(x',y')) = d_1(x,x') + d_2(y,y')$ 1-norm 

$d((x,y),(x',y')) = \sqrt{d_1(x,x')^2 + d_2(y,y')^2}$ 2-norm 

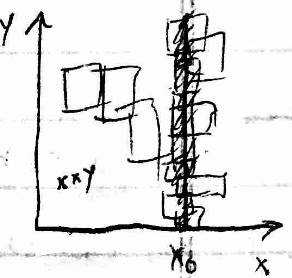
$d((x,y),(x',y')) = \max(d_1(x,x'), d_2(y,y'))$. sup-norm 

THM. If X, Y compact, then $X \times Y$ compact.

Proof. Let $\{W_\alpha\}$ be an open cover of $X \times Y$.

Lemma. WLOG, each W_α is of the form:

$W_\alpha = U_\alpha \times V_\alpha$, where U_α is open in X and V_α is open in Y.



Indeed, each W_α is a union of squares so consider the cover of $X \times Y$ by all these squares. If a finite subcover is found using these squares, then it clearly defines a finite subcover using the original W_α's.

Claim. $X \times Y$ covered by $U_\alpha \times V_\alpha$, then for every $x_0 \in X$, we can find $\epsilon > 0$ such that $U(x_0;\epsilon) \times Y$ is covered by finitely many $U_\alpha \times V_\alpha$'s.

Proof. By compactness of Y, there exists finite set F s.t. $\bigcup_{\alpha \in F} U_\alpha \times V_\alpha \supseteq \{x_0\} \times Y$.

But, $\bigcup_{\alpha \in F} U_\alpha \times V_\alpha \supseteq \bigcup_{\alpha \in F} (\bigcap_{\beta \in F} U_\beta) \times V_\alpha \supseteq (\bigcap_{\beta \in F} U_\beta) \times Y$

open since finite.

Lemma. If X, Y compact, so is $X \times Y$.

Lemma² If $W_\alpha = U_\alpha \times V_\alpha$ is an open cover of $X \times Y$, then every $x \in X$ has an open nbd Z_x such that $Z_x \times Y$ can be covered by finitely many W_α's.

Lemma 2. A closed subset A of compact X is compact.

THM. If $f: X \to Y$ is continuous, X is compact, then $f(X)$ is compact.

Proof. If $f(X) \subseteq \bigcup V_\alpha$ where each V_α is open in Y, then $X \subseteq \bigcup f^{-1}(V_\alpha)$ is an open cover of X by $f^{-1}(V_\alpha)$'s, which are open by cont. of f.
So, by compactness of X, one can find $\alpha_1, \dots, \alpha_n$ s.t.
$\bigcup_i f^{-1}(V_{\alpha_i})$ covers X and then $\{V_{\alpha_i}\}$ covers $f(X)$.
This means $f(X)$ is also compact, since every open cover V_α has a finite subcover $\{V_{\alpha_i}\}$.

MAXIMAL VALUE THM

Corollary. A continuous function $f: X \to \mathbb{R}$ where X is compact is bounded, and it attains its bounds: $\exists x_0 \in X$ s.t. $f(y) \leq f(x_0)$ for all $y \in X$.

Proof. By previous THM, $f(X)$ is compact.
But $f(X) \subseteq \mathbb{R}$ so it is closed and bounded.
Therefore, f is bounded and $f(X)$ is closed, meaning it contains its limit points, so the supremum $\sup f(x) \in f(X)$.
$\Rightarrow \sup_X f(x) = f(x_0)$ for some x_0.
Therefore, for all $y \in X$, $f(y) \leq f(x_0)$.

Definition. A space X is connected if there are no clopen sets in X (other than $\emptyset$, X).

THM 1. $A \subseteq \mathbb{R}$ connected IFF it is a generalized interval.

All poss: $[a,b)$, $(a,b]$, (a,b), $[a,b]$, $(-\infty,a]$, $(-\infty,b)$, (b,∞), $[a,\infty)$.

THM 2. If X, Y connected, then $X \times Y$ connected.

THM 3. If X is connected, $f: X \to Y$ is continuous, $f(X)$ is connected.

$\{\forall a,b \in X \mid [a,b] \in X\} \Leftrightarrow X$ is convex

$\longrightarrow$

INTERMEDIATE VALUE THM.

Corollary. Let X be connected. If $f: X \to Y$ is continuous, then $f(X)$ is a connected subspace of Y. In particular, if $f: X \to \mathbb{R}$ is continuous and if $f(x_0) < r < f(x_1)$ for some pts. x_0, x_1 of X, then $f(x) = r$ for some x of X.

Proof. Suppose $f(X) = A \cup B$, where A, B disjoint sets open in $f(X)$. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoints sets whose union is X, each of which is open since f is continuous. This contradicts connectedness of X.

Given f, let A consist of all $y \in \mathbb{R}$ with $y < r$ and B consist of all $y \in \mathbb{R}$ with $y > r$.

Then A, B open in $\mathbb{R}$; if $f(X)$ does not contain r, then $f(X)$ is the union of disjoint sets $f(X) \cap A$ and $f(X) \cap B$, each of which is open in $f(X)$. This contradicts connectedness of $f(X)$.