PSet 1: Partial Solutions

DISCLAIMER: I cannot claim that what I have written here constitutes a perfect solution. Certainly some mistakes are present; hopefully these mistakes aren't too severe. I hope that my answers may serve as a guide to you when studying for the final exam.

Problem §1 Q1a

If $x = 0$, then $\langle 0, y\rangle = 0\langle 0, y\rangle = 0 = 0\|y\| = \|x\|\|y\|$. Similarly, if $y = 0$, then $\langle x, 0\rangle = 0\langle x, 0\rangle = 0 = \|x\|0 = \|x\|\|y\|$. Otherwise, $x, y \neq 0$, and so we may define $c = 1/\|x\|$ and $d = 1/\|y\|$. By properties of the inner product,

$$\begin{aligned}
0 &\leq \langle cx - dy, cx - dy\rangle \\
&= \langle cx, cx - dy\rangle - \langle dy, cx - dy\rangle \\
&= \langle cx, cx\rangle - \langle cx, dy\rangle - \langle dy, cx\rangle + \langle dy, dy\rangle \\
&= \|cx\| - \langle cx, dy\rangle - \langle cx, dy\rangle + \|dy\| \\
&= \|\frac{1}{\|x\|}x\| - 2\langle cx, dy\rangle + \|\frac{1}{\|y\|}y\| \qquad (1) \\
2\langle cx, dy\rangle &\leq \frac{1}{\|x\|}\|x\| + \frac{1}{\|y\|}\|y\| \\
\frac{1}{\|x\|}\frac{1}{\|y\|}\langle x, y\rangle &\leq 1 \\
\langle x, y\rangle &\leq \|x\|\|y\|
\end{aligned}$$

Thus the claim is true for all $x, y \in \mathbb{R}$. □

Problem §2 Q6

Attached.

Problem §3 Q4c

We claim that, in general, if U is open then $U \subseteq int(\bar{U})$, and moreover that in general $U \not\supset int(\bar{U})$.

Proof. We first prove that $U \not\supset int(\bar{U})$. Consider the following counter-example to the claim that $U \supset int(\bar{U})$: Consider the metric space $X = \mathbb{R}$, let $U = \mathbb{R} \setminus \{0\}$, and consider the point $0 \in \mathbb{R}$. Then we also have that $0 \in lp(U)$ since for all $\epsilon > 0$ the point $\epsilon/2$ is also contained in the ϵ-neighbourhood $U(x, \epsilon)$. Therefore, $\bar{U} = U \cup lp(U) = \mathbb{R}$, and so $\bar{U}$ is open. Before continuing, we must first prove the following assertion:

Subclaim: If $T \subseteq X$ and T is open, where X is a metric space, then $T = int(T)$. Suppose that $t \in T$. Since T is open, then there exists $\epsilon > 0$ such that $U = (t, \epsilon) \subseteq T$. But this implies, by definition of $int(T)$ as the set of all points in T that have an ϵ-ball contained in T, that $t \in int(T)$, so that $T \subseteq int(T)$. Conversely, it follows by definition that $T \supseteq int(T)$. Therefore, $T = int(T)$ whenever T is open.

Since $\bar{U}$ is open, it thus follows by our subclaim that $\bar{U} = int(\bar{U})$, and so $int(\bar{U}) = \mathbb{R}$. But $\mathbb{R}$ contains the point 0, a contradiction. Therefore, $U \not\supset int(\bar{U})$.

Next we show that $U \subseteq int(\bar{U})$. Suppose that $x \in U$. Since $x \in U$, then there exists $\epsilon > 0$ such that $U(x, \epsilon) \subseteq U \subseteq \bar{U}$. Thus, by definition of $int(\bar{U})$ as the set of all points in $\bar{U}$ that have an ϵ-ball contained in $\bar{U}$, $x \in int(\bar{U})$, which implies that $U \subseteq int(\bar{U})$. ■

Problem §3 Q6

For $f : X \to Y$ to be continuous, it is sufficient to show that if $V \subseteq Y$ is closed, then $f^{-1}(Y) = \{x | f(x) \in Y\}$ is closed in X. So, let $V \subseteq Y$. We first claim that $f^{-1}(V) = f|_A^{-1}(V) \cup f|_B^{-1}(V)$.

Proof. We first show that $f^{-1}(V) \subseteq f|_A^{-1}(V) \cup f|_B^{-1}(V)$. Suppose that $x \in f^{-1}(V)$. Since A and B cover X, then $x \in A$ or $x \in B$. If $x \in A$, since $x \in f^{-1}(V)$ then $x \in f|_A^{-1}(V)$. Similarly, if $x \in B$, since $x \in f^{-1}(V)$ then $x \in f|_B^{-1}(V)$. In either case, $x \in f|_A^{-1}(V) \cup f|_B^{-1}(V)$, so that $f^{-1}(V) \subseteq f|_A^{-1}(V) \cup f|_B^{-1}(V)$.

For inclusion in the other direction, consider $x \in f|_A^{-1}(V) \cup f|_B^{-1}(V)$. Then $x \in f|_A^{-1}(V)$ or $x \in f|_B^{-1}(V)$. In the former case, this implies that $x \in A$ and $f(x) \in V \Rightarrow x \in X$ and $f(x) \in V \Rightarrow x \in f^{-1}(V)$. The case where $x \in B$ is handled identically. Therefore, $f|_A^{-1}(V) \cup f|_B^{-1}(V) \subseteq f^{-1}(V)$, which gives us our conclusion: $f|_A^{-1}(V) \cup f|_B^{-1}(V) = f^{-1}(V)$. ■

Continuing with our proof of the problem statement, by the continuity of f_A and f_B, we have by V being closed that $f_A^{-1}(V)$ is closed in $A \subseteq X$, and $f_B^{-1}(V)$ is closed in $B \subseteq X$. But by Theorem 3.2, this implies that there exist closed subsets $S_A, S_B \subseteq X$ such that $f_A^{-1}(V) = S_A \cap A$, and $f_B^{-1}(V) = S_B \cap B$. But arbitrary intersections of closed sets are closed, so therefore $f_A^{-1}(V)$ and $f_B^{-1}(V)$ are closed. Thus, the finite union of closed sets $f^{-1}(V) = f_A^{-1}(V) \cup f_B^{-1}(V)$ is also closed, and so f is continuous, as required.

Problem §3 Q9e

We claim that $int(A) = \emptyset = ext(A)$, and $Bd(A) = \mathbb{R}^2$.

Proof. Suppose that $a \in A$. Then $a = (x, y)$ for some rational x, y. Suppose for contradiction that $a \in U$ for some open subset $U \subseteq A$. Since $A \subseteq \mathbb{R}^2$, we assume our metric is either the Euclidean-norm or the sup-norm. Then, by the definition of openness, there must exist some $\epsilon > 0$ such that $U(a, \epsilon) \subseteq A$. If $\epsilon \in \bar{\mathbb{Q}}$ then the point $(x, y + \epsilon)$, for which $|(x, y) - (x, y + \epsilon/2)| \leq \|(x, y) - (x, y + \epsilon/2)\| = \|(0, -\epsilon/2)\| = \epsilon/2 < \epsilon$, is contained in $U(a, \epsilon)$ but not in A, a contradiction. Otherwise, if $\epsilon \in \mathbb{Q}$, then by a similar argument we have that the point $(x, y + \epsilon/\sqrt{2})$ is in $U(a, \epsilon)$ but not in A, a contradiction. Therefore, $int(A) = \emptyset$.

Subclaim: for any $c, d \in \mathbb{R}$ such that $c < d$, $\exists r \in \mathbb{Q}$ such that $c < r < d$. Proof: Let $z = d - c > 0$. By the Archimedean property of the real numbers, $\exists n \in \mathbb{N}$ such that $n > 1/z \Rightarrow nz > 1 \Rightarrow nd - nc > 1$. But this implies that $\exists m \in \mathbb{N}$ such that $nc < m < nd \Rightarrow c < \frac{m}{n} = r < d$.

Now suppose that $a \in A^c$. Then $a = (x, y)$ where either x is irrational or y is irrational. Now, suppose for contradiction that $a \in U$ for some open subset $U \subseteq A^c$. Then $\exists \epsilon > 0$ such that $U(a, \epsilon) \subseteq U \subseteq A^c$. Consider the point $(x + \epsilon/2, y + \epsilon/2) \in \mathbb{R}$. By the subclaim proved above, there exist rational numbers r_1, r_2 which satisfy $x < r_1 < x + \epsilon/2$ and $y < r_2 < y + \epsilon/2$. But then $|(x, y) - (r_1, r_2)| \leq \|(x, y) - (r_1, r_2)\| = \|(x - r_1, y - r_2)\| =$

$\sqrt{(x-r_1)^2+(y-r_2)^2} \leq \sqrt{(x-(x+\epsilon/2))^2+(y-(y+\epsilon/2))^2} = \sqrt{\epsilon^2/2} < \epsilon$. Hence, the point $(r_1, r_2) \in U(a, \epsilon)$, but $(r_1, r_2) \notin A^c$, a contradiction. Thus, $ext(A) = int(A^c) = \emptyset$.

It follows that $Bd(A) = \mathbb{R}^2 \setminus (int(A) \cup ext(A)) = \mathbb{R}^2$. $\square$