$\mathcal{F}(S,\mathbb{R}) = \{ f : S \to \mathbb{R} \}$ $\quad f(s) = 0 \;\; \forall s \in S. \quad f \in \mathcal{F}(S,\mathbb{R})$.

row. e. form is unique where e. fors is not.

Lecture 16.10.06

$2x - 7y = -3$
$-3x + 2y = -4$

$$m \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right\} \text{system of lin. eq'ns}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

$\boxed{Ax = b} \iff$ our system

$x - y = 7$
$x + 3y = 2$

$x \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} \quad b \longmapsto \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

Lucky Case: A is square & invertible

our system

$$Ax = b \iff \underbrace{A^{-1}A}_{} x = A^{-1} \cdot b$$
$$I x = A^{-1} \cdot b$$
$$\Rightarrow \boxed{x = A^{-1} \cdot b} \text{ lucky case}$$

Theory:

$b=0$ "homogeneous system"

Claim: Consider $Ax=0$ $\qquad A \in M_{n\times m}(F)$ also a lin. trans.

① $X=0$ is always a solution $\qquad A: F^n \longrightarrow F^m$

② X is a solution iff $X \in$ null space (A)
$= \text{kernel}(A)$: $Ker(A)$

$b \neq 0$ "non-homogeneous system"

claim: Consider $Ax=b$

① System has a solution iff $b \in \text{range}(A) = Im\, A$

② If X_0 solves the system $(AX_0=b)$

X_1 solves system iff $X_1 = X_0 + X$ where X solves the homogeneous part of the matrix. $\Rightarrow \boxed{AX=0}$

Proof: We know $AX_0=b$

$$AX_1=b \iff A(X_0+X)=b \iff AX_0+AX=b$$

$$\iff b+AX=b \iff AX=0$$

$X_1 = X_0 + X$ (homogeneous part of the matrix)

$X = X - X_0$

We knew

row operations = multiplying elem. matrices on the left

With row reduction:

$$A \rightsquigarrow A' = \underset{\sim}{E} \cdot A \quad \text{"in r.r.e.f"}$$

a product of invert. matrices, hence invertible.

$$Ax = b \iff \underbrace{EA}_{A'}x = \underbrace{Eb}_{b'} \iff A'x = b'$$

$$\Updownarrow$$

$$T(Ax) = T(b)$$

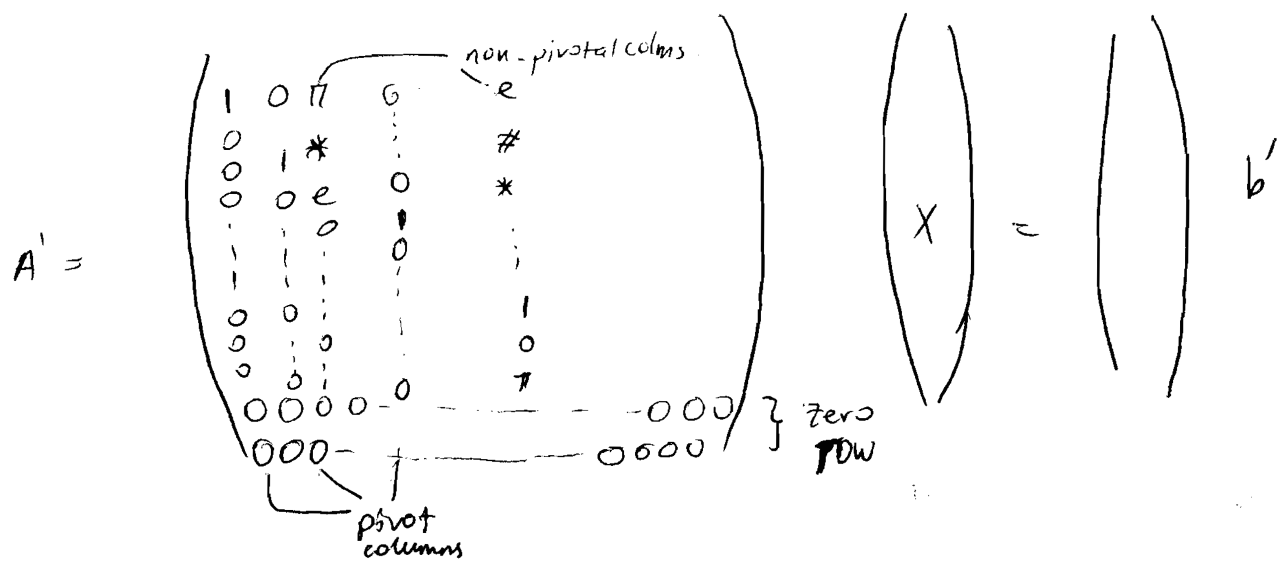
Mechanical Aside: To find A' & b' at the same time,

form $\tilde{A} = (A/b)$ row-reduce it

$$E\tilde{A} = (\underbrace{EA}_{A'} / \underbrace{Eb}_{b'})$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 7 \\ 2 \end{pmatrix} \quad \tilde{A} = \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 1 & -1 & 2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & -2 & -5 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & 1 & 5/2 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 9/2 \\ 0 & 1 & 5/2 \end{array}\right)$$



Thm: ① If b' has a non-zero entry in one of the zero rows of A', no solution.

② Otherwise all solutions are obtained by setting the entries of x corresponding to the non-pivotal col's in an arbitrary way and then the rest is uniquely determined by solving single lin. eqn's.

Tutorial:

Defn: $Ax=b$, $b, x \in V^n$, $A \in M_{n,n}(V)$ if $b=0$, then this system is called homogeneous.

Thm: If the system $T_A x = 0$ then is homog. then the dimention k of the space of sol's. of that system

$k = n - \operatorname{rank} T_A = n - \operatorname{rank} A$

dimension of the kernel of T_A (nullity of T_A)

• max # of lin. indep. columns in $A = \operatorname{rank}(T_A)$ = max # of lin. indep. rows in A.

1) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$T(a,b,c) = (a+b, 2a-c)$

Determine $T^{-1}(1,11)$ i.e. the set of preimages of the point $(1,11)$

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Tutorial:

Defn: $Ax=b$, $b, x \in V^n$, $A \in M_{n,n}(V)$ if $b=0$, then this system is called homogeneous.

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$\downarrow$ dimension of the kernel of T_A (nullity of T_A)

• max # of lin. indep. columns in $A = \text{rank}(T_A)$ = max # of lin. indep. rows in A.

1) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$T(a,b,c) = (a+b, 2a-c)$

Determine $T^{-1}(\{1,1\})$ i.e. the set of preimages of the point $(1,1)$

Sol'n: We want (a,b,c) s.t. $T(a,b,c) = (a+b,\ 2a-c) = (1,1)$

or $\begin{cases} a+b = 1 \\ 2a-c = 11 \end{cases} \iff \begin{cases} a+b+0\cdot c = 1 \\ 2a+0b-c = 11 \end{cases}$

The corresponding augmentent matrix is.

$$A := \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & 11 \end{array}\right) \qquad \operatorname{rank} A = 2$$

while $\operatorname{rank}\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix} = 2$ Then solution exists.

Remark: Assume $Ax=b$. This means that $b \in R(A)$. If $\{a_1, \dots a_k\}$ are all the lin. indep. columns of A then $\operatorname{rank} A = k$. The sol'n of this system exists. iff $b \in R(A)$, i.e. $b \in \operatorname{span}\{a_1, \dots a_k\}$

$$\iff k = \dim R(T) = \dim \operatorname{span}\{a_1, \dots a_k\} = \underbrace{\dim \operatorname{span}\{a_1, \dots a_k, b\}}_{= \operatorname{rank}(A/b)} = k$$

Example: $\begin{cases} x_1 + x_2 = 2 \\ x_1 + x_2 = 3 \end{cases}$

Aug. matrix $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$ has rank 2, while non-augmented matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has rank 1 back to prev. example

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & 11 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 0 & -1 & 11 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 0 & -1 & 11 \\ 0 & 2 & 1 & -9 \end{array}\right)$$

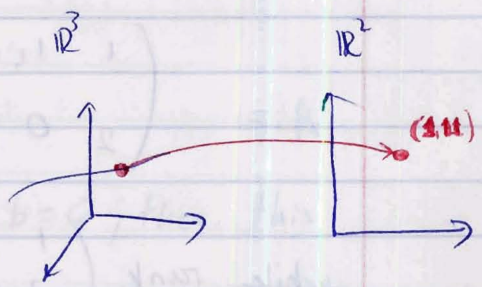
then

$$2a \quad - c = 11$$

$$2b + c = -9$$

set $c = t \Rightarrow b = \frac{-9-t}{2}, \; a = \frac{11+t}{2}$

Conclusion: $T^{-1}(1,11) = \left\{ \frac{11+t}{2}, \frac{-9-t}{2}, c \right\} \rightarrow$



(2) Determine if the system

$$x_1 + 2x_2 + 3x_3 = 1$$

$$x_1 + x_2 - x_3 = 0$$

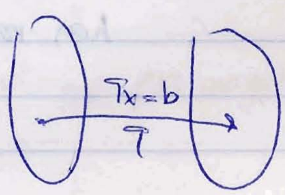
$$x_1 + 2x_2 + x_3 = 3$$

Soln: Non-augmented matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} =: A$

The rank of A

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & -2 \end{pmatrix} \Rightarrow \text{rank } A = 3$$

In this case we conclude that the system is consistent so we don't have to determine the rank of augmented matrix



3) — " —— $x_1 + x_2 - x_3 = 1$

$2x_1 + x_2 + 3x_3 = 2$

Augmented matrix

$(A|b)$ $\begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & 1 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -1 & 5 & 0 \end{pmatrix}$

so the rank $(A|b) = 2$

rank $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 3 \end{pmatrix} =$ rank $\begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 5 \end{pmatrix} = 2 \Rightarrow$ system has a solⁿ.

Reduced row echelon form:

$x_1 + 2x_2 - x_3 = 1$

$2x_1 + 2x_2 + x_3 = 1$ Solve by Gauss method.

$3x_1 + 5x_2 - 2x_3 = -1$

<u>Solⁿ</u> $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -2 & 3 & -1 \\ 0 & -1 & 1 & -4 \end{array}\right)$

$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & +4 \\ 0 & -2 & 3 & -1 \end{array}\right) \sim \underbrace{\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 7 \end{array}\right)}_{\text{row-echelon form}}$ continue to make '0' to reach to r.e.r. form

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 8 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -14 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 7 \end{array}\right)$$

$$\sim \quad x_1 = -14$$
$$x_2 = 11$$
$$x_3 = 7$$

Example: Let $W \subset M_{2\times 2}(\mathbb{R})$ consisting of the set symmetric matrix

$W = Symm_{2\times 2}(\mathbb{R})$

Consider

$$W \supset S = \left\{ \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 9 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \right\}$$

$$\text{span} \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle = W$$

Find the basis of W in S

Soln: We have to find 3 lin. indp. matrices in S.

To this end let us realize each 2×2-matrix as a vector in $\mathbb{R}^4$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow (a\ b\ c\ d)$$

Generate a matrix

$$A \sim \begin{pmatrix} 0 & 1 & 2 & 1 & 1 \\ -1 & 2 & 1 & -2 & 2 \\ -1 & 2 & 1 & -2 & 2 \\ 1 & 3 & 9 & 4 & -1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 2 & 1 & -1 \\ 1 & 3 & 9 & 4 & -1 \\ -1 & 2 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 9 & 4 & -1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 5 & 10 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 9 & 4 & -1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 9 & 4 & -1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 3 & 9 & 4 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 9 & 4 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 9 & 0 & 7 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Conclusion: rank $A = 3$ &

$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ generate $R(T_A)$ and then

$\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ are a basis of W