MAT257 Midterm 1 Review — TOPOLOGY

Metric Spaces: Given a set X, a metric on X is a function $d: X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:

COMMUTATIVITY ① $d(x,y) = d(y,x)$

POSITIVITY ② $d(x,y) \geq 0$ AND $d(x,y) = 0$ IFF $x = y$.

TRIANGLE INEQUALITY ③ $d(x,z) \leq d(x,y) + d(y,z)$

A metric space is a set X together with a specific choice of metric on X.

If X is a metric space with metric d and $Y \subseteq X$, then d restricted to $Y \times Y$ is a metric on Y; so Y is a metric subspace of X.

Ex: [1] In $\mathbb{R}^n$, there are two main matrices:

EUCLIDEAN METRIC $d_1(x,y) = \| x - y \| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

SUP METRIC $d_2(x,y) = | x - y | = \max_{1 \leq i \leq n} |x_i - y_i|$

[2] In $C([0,1]) = \left\{ \begin{array}{l} \text{continuous functions,} \\ f:[0,1] \to \mathbb{R} \end{array} \right.$ with d as follows:

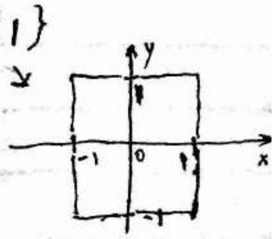
for $f, g: [0,1] \to \mathbb{R}$ continuous, define $d(f,g) = \max_{x \in [0,1]} |f(x) - g(x)|$.

Definition. Given a metric space X and $x_0 \in X$ and $\epsilon > 0$ (given),

$U(x_0, \epsilon) = \{ x \in X : d(x_0, x) < \epsilon \}$: ϵ-nbd/ball centred at x_0.

Ex: Given $X = \mathbb{R}^2$ and $d_2 = |\cdot|_{sup}$; $U(0;1) = \{ x \in \mathbb{R}^2 : d(0,x) < 1 \}$

If $x = (a,b)$, $d(x,0) = |x - 0| = |x| = \max\{|a|, |b|\} \leq 1$



Definition. A set $U \subset X$ (X is a metric space) is called <u>open</u> if $\forall x_0 \in U.\ \exists \epsilon > 0 : U(x_0, \epsilon) \subset U$.

A set $F \subset X$ is <u>closed</u> if $F^c = X \setminus F$ is open.

<u>Claim.</u> $U(x_0, r)$ is open.

<u>Proof.</u> Let $y \in U(x_0, r)$ and take $\epsilon = r - d(x_0, y) > 0$, since $y \in U(x_0, r)$ so $d(x_0, y) < r$.

<u>Sub-claim.</u> $U(y, \epsilon) \subset U(x_0, r)$.

<u>Proof.</u> Let $z \in U(y, \epsilon)$, i.e. $d(z, y) < \epsilon$.

$$\begin{aligned} \text{Then} \quad d(x_0, z) &\le d(x_0, y) + d(y, z) \\ &< d(x_0, y) + \epsilon \\ &= \cancel{d(x_0, y)} + r - \cancel{d(x_0, y)} = r. \end{aligned}$$

So, $d(x_0, z) < r \Rightarrow z \in U(x_0, r)$.

Therefore, $U(x_0, r)$ is open.

<u>THM 1a.</u> ① $\emptyset, X$ are open

② $\forall \alpha \in I$, U_α open $\Longrightarrow \bigcup_{\alpha \in I} U_\alpha$ is open.

③ $\forall 1 \le i \le n$, U_i open $\Longrightarrow \bigcap_{i=1}^{n} U_i$ is open.

<u>THM 1b.</u> ① $\emptyset, X$ are closed

② $\forall \alpha \in I$, F_α closed $\Longrightarrow \bigcap_{\alpha \in I} F_\alpha$ is closed.

③ $\forall 1 \le i \le n$, F_i closed $\Rightarrow \bigcup_{i=1}^{n} F_i$ is closed.

<u>OPEN:</u> ↓
$\emptyset, X$;
arbitrary UNIONS;
finite INTERSECTIONS.

$\emptyset, X$
arbitrary INTERSECTIONS
finite UNIONS
<u>CLOSED</u>: ↑

<u>Proof of THM 1a.</u> ① $\emptyset, X$ are open trivially.

② Assume $\forall \alpha \in I$, U_α open. Let $x \in \bigcup_{\alpha \in I} U_\alpha$, i.e. $\exists \alpha_0 \in I$ such that $x \in U_{\alpha_0}$. Then, let $\epsilon > 0$ be such that $U(x, \epsilon) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$ is open.

③ Let $x \in \bigcap_{i=1}^{n} U_i \Rightarrow \forall i,\ x \in U_i$. For each i, choose ϵ_i s.t. $U(x, \epsilon_i) \subseteq U_i$. Let $\epsilon = \min_{1 \le i \le n} \epsilon_i > 0$ and $U(x, \epsilon) \subseteq U(x, \epsilon_i) \subseteq U_i \Longrightarrow U(x, \epsilon) \subseteq \bigcap_{i=1}^{n} U_i$ is open.

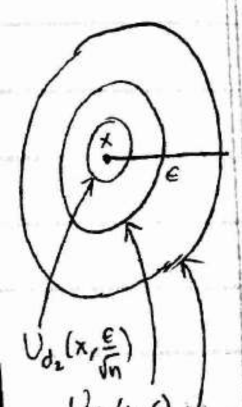
THM 1b can be proved w/ De Morgan's Laws on ② and ③.

OPEN/CLOSED rel. to d_1, d_2.

Given $\mathbb{R}^n$ and $\|\cdot\|$ i.e. $d_1(x,y) = \|x-y\|$ and $d_2(x,y) = |x-y|$

<u>THM 4a.</u> $\mathcal{U}$ is open relative to $d_1 \iff \mathcal{U}$ is open relative to d_2.

<u>THM 4b.</u> F is closed relative to $d_1 \iff F$ is closed relative to d_2.



<u>Proof</u> of THM 4a ($\Rightarrow$): Assume $\mathcal{U}$ is d_1-open. Let $x \in \mathcal{U}$. Then,

$(d_2 \le d_1 \le \sqrt{n}\, d_2)$ $\quad \exists \epsilon > 0$ such that $U_{d_1}(x, \epsilon) \subseteq \mathcal{U}$.

<u>Claim.</u> $U_{d_2}(x,\epsilon) \supseteq U_{d_1}(x,\epsilon) \supseteq U_{d_2}(x, \frac{\epsilon}{\sqrt{n}})$.

<u>Proof.</u> If $y \in U_{d_1}(x,\epsilon)$, then: $\qquad \therefore U_{d_2}(x,\epsilon) \supseteq U_{d_1}(x,\epsilon)$.

$d_2(x,y) \le d_1(x,y) < \epsilon$, so $y \in U_{d_2}(x,\epsilon)$.

If $y \in U_{d_2}(x, \frac{\epsilon}{\sqrt{n}})$, then: $\qquad \mathcal{U} \supseteq U_{d_1}(x,\epsilon) \supseteq U_{d_2}(x, \frac{\epsilon}{\sqrt{n}})$

$d_2(x,y) < \frac{\epsilon}{\sqrt{n}} \Rightarrow \sqrt{n}\, d_2(x,y) < \epsilon$.

But, $d_1(x,y) \le \sqrt{n}\, d_2(x,y) < \epsilon$, so $y \in U_{d_1}(x,\epsilon)$.

Therefore, $\mathcal{U}$ is d_2 open.

($\Leftarrow$) can be proved using the fact that $(\frac{1}{\sqrt{n}} d_1 \le d_2 \le d_1)$.

(4b) can be proving using (4a).

SUB METRIC SPACE

<u>THM</u> X metric space, $Y \subseteq X$ is also a metric space.

[1] $U \subset Y$ is open IFF $\exists V \subseteq X$ such that V is open in X and $U = V \cap Y$.

[2] $F \subset Y$ is closed IFF $\exists G \subseteq X$ such that G is closed in X and $F = G \cap Y$.

CLOPEN

<u>Note.</u> All subsets of finite metric spaces are clopen.

LIMIT POINT

<u>Definition.</u> A point $x_0 \in X$ is called a <u>limit point</u> of some subset $A \subseteq X$ if:

$\forall \epsilon > 0. \left((U(x_0,\epsilon) \cap A) \setminus \{x_0\} \ne \emptyset \right)$. (i.e. $\forall \epsilon > 0.\ |U(x_0,\epsilon) \cap A| = \infty$).

CLOSURE

<u>Definition.</u> The closure $\bar{A}$ of set A; $\bar{A} := A \cup lp(A)$.

<u>THM.</u> A set A is closed IFF $A = \bar{A}$. $\rightarrow$

THM. A is closed $\Leftrightarrow$ $A = \bar{A}$.

Proof. A^c open $\Leftrightarrow$ $lp(A) \subset A$.

($\Rightarrow$) Suppose by contradiction that $x_0 \in lp(A) \cap A^c$.
Find $\epsilon > 0$ such that $U(x_0, \epsilon) \subset A^c$, but
$U(x_0, \epsilon) \cap A \neq \phi$, which will contradict $x_0 \in lp(A)$.

($\Leftarrow$) If $A = \bar{A}$, the $A = \bar{A}$ is closed, immediately.

CLOSURE PROPERTIES.

Note: $\bar{A}$ is the smallest closed set contained in A.
$\bar{A}$ is the intersection of all closed sets containing A.

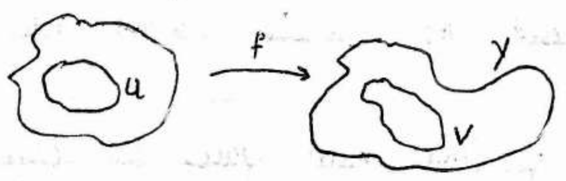
Definition. Let $f: X \to Y$ (X has metric d_x and Y has metric d_y).
Let $x_0 \in X$. We say that f is continuous at x_0 if

CONTINUITY

for every nbd V of $f(x_0)$, there exists a nbd $\mathcal{U}$ of x_0 s.t. $f(\mathcal{U}) \subset V$.
$(\forall \epsilon > 0.\ \exists \delta > 0:\ d_x(x_0, x) < \delta \Rightarrow d_y(f(x_0), f(x)) < \epsilon)$.

THM (TFAE): ① f is continuous
② For every open set $V \subset Y$, $f^{-1}(V)$ is open.
③ For every closed set $F \subset Y$, $f^{-1}(F)$ is closed.

$f: X \to Y$



$f^{-1}(V) = \{x \in X : f(x) \in V\}$.
$f(\mathcal{U}) = \{f(u) : u \in \mathcal{U}\}$.

Claim. Given $f: X \to Y$ and $A, B \subseteq X$; $C, D \subseteq Y$.

[1] $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$, $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

[2] $f^{-1}(D^c) = (f^{-1}(D))^c$

[3] $f(A \cup B) \supseteq f(A) \cup f(B)$, $f(A \cap B) \subseteq f(A) \cap f(B)$

[4] $f(A^c) \supseteq (f(A))^c$

Proof of THM (TFAE).

$(1 \Rightarrow 2)$. Let $V \subset Y$ open and let $x_0 \in f^{-1}(V)$, i.e. $f(x_0) \in V$.

So, V is a nbd of $f(x_0)$ and by the continuity assumption, there is a nbd of x_0, U, s.t. $f(U) \subseteq V$. But $x_0 \in U \subseteq f^{-1}(V)$.

So, $f^{-1}(V)$ is open.

$(2 \Rightarrow 1)$ Let $x_0 \in X$. Let V be a nbd of $f(x_0)$.

Let $U = f^{-1}(V)$, which is open by assumption.

Clearly, $x_0 \in U$ so U is a nbd of x_0 and $f(U) = f(f^{-1}(V)) \subseteq V$, as req.

$(2 \Rightarrow 3)$ $\underbrace{f^{-1}(F^c)}_{\text{open}} = (\underbrace{f^{-1}(F)}_{\text{closed}})^c$ $(3 \Rightarrow 2)$ Same way.

THM. (Technical)

1. Constant functions are continuous.
2. $I: X \to X$ is continuous.
3. $f: X \to Y$ continuous $\Rightarrow$ $f|_A : A \to Y$ continuous.
4. $f: X \to Y$, $g: Y \to Z$ both continuous $\Rightarrow$ $g \circ f$ continuous.
5. $f: X \to \mathbb{R}^n$, $f = (f_1, f_2, \dots f_n)$ continuous $\iff$ f_i continuous for all $i = 1, \dots, n$.
6. $\pi_i : \mathbb{R}^n \to \mathbb{R}$, $\pi_i(x) = x_i$ is continuous.
7. $f, g : X \to \mathbb{R}$ continuous $\Rightarrow$ $f+g$, $f \cdot g$, $f-g$, $f \div g$ $(g \neq 0)$ continuous.

INT Definition. If $A \subseteq X$ is a subset, $\operatorname{int} A = \{x \in A : \exists \varepsilon > 0.\ U(x, \varepsilon) \subseteq A\}$.
$=$ maximal open set cont. in A = union of all open sets contained in A.

EXT Definition. If $A \subset X$ is a subset, $\operatorname{ext} A = \operatorname{int}(A^c)$

BD Definition. If $A \subset X$ is a subset, $\operatorname{bd} A = X \setminus (\operatorname{int}(A) \cup \operatorname{ext}(A))$.

Claims. 1. $\operatorname{ext} A = X \setminus \bar{A}$ 2. $\operatorname{int} A = X \setminus \overline{A^c}$ 3. $\operatorname{bd} A = \bar{A} \cap \overline{A^c}$

CLOSED RECTANGLE — **Definition.** Q is the closed rectangle : $\underline{Q} = [a_1, b_1] \times \ldots \times [a_n, b_n]$,

consisting of all points x of $\mathbb{R}^n$ s.t. $a_i \leq x_i \leq b_i$ for all i.

OPEN RECTANGLE — Then, $\underline{\text{int } Q} = (a_1, b_1) \times \ldots \times (a_n, b_n)$, is the open rectangle,

consisting of all points x of $\mathbb{R}^n$ s.t. $a_i < x_i < b_i$ for all i.

$\underline{\text{ext } Q} = \mathbb{R}^n - Q$

$\underline{\text{bd } Q} = Q - \text{int } Q$

CLOSED CUBE — **Definition.** The closed cube centred at a (or simple cube centred at a):

$$C = [a_1 - \epsilon, a_1 + \epsilon] \times \ldots \times [a_n - \epsilon, a_n + \epsilon].$$

OPEN CUBE. — The open cube centred at a:

$$C(a;\epsilon) = (a_1 - \epsilon, a_1 + \epsilon) \times \ldots \times (a_n - \epsilon, a_n + \epsilon).$$