Apr. 3rd.

1. Poincaré's Lemma:

If $d\omega = 0$, then $\exists \lambda$ $d\lambda = \omega$.

2. $\int d(\omega \wedge \eta) = \int d\omega \wedge \eta + (-1)^{\deg(\omega)} \int \omega \wedge \eta$

$(-1)^{\deg(\omega)} \int d(\omega \wedge \eta) = (-1)^{\deg(\omega)} \int d\omega \wedge \eta + \int \omega \wedge d\eta$.

$\int_M \omega \wedge d\eta = (-1)^{\deg(\omega)} \int_M d(\omega \wedge \eta) - (-1)^{\deg(\omega)} \int d\omega \wedge \eta$

$= (-1)^{\deg(\omega)} \int_{\partial M} \omega \wedge \eta - (-1)^{\deg(\omega)} \int_M d\omega \wedge \eta$.

if $\partial M = \phi$

or support $\omega \cap \partial M = \phi$ $\quad = -(-1)^{\deg(\omega)} \int_M d\omega \wedge \eta$.

$\dim \Lambda^k(\mathbb{R}^n) = \binom{n}{k} = \binom{n}{n-k} = \dim \Lambda^{n-k}(\mathbb{R}^n)$

$*: \Lambda^k \to \Lambda^{n-k}$ $\quad *(\phi_I) = \pm \phi_{I^c}$. $\quad \mathbb{R}^3: *(\phi_1) = \phi_{23}$

$\phi_I \wedge *(\phi_I) = \phi_n$

on $\mathbb{R}^4_{txyz}$ $\quad *(dt) = S\, dx \wedge dy \wedge dz$. $\quad S = 1$

$dt \wedge dx \wedge dy \wedge dz = dtxyz$.

$*(dx\,dy\,dz) = S\, dt$ ~~$= dt$~~

$dx\,dy\,dz \wedge (S\, dt) = dt\,dxyz \Rightarrow S = -1$.

3. Hodge $*: \Omega^k(\mathbb{R}^n) \to \Omega^{n-k}(\mathbb{R}^n)$ s.t.

if $\omega, \eta \in \Omega^k(\mathbb{R}^n)$

$\langle \omega, \eta \rangle dx_I \quad = \quad \omega \wedge (*\eta).$

$\sum f_I dx_I \quad \sum g_I dx_I \qquad dx_I \quad dx_J.$

$\sum f_I g_I \qquad \delta_{IJ}\, dx_I$

Maxwell's equ. $\Longleftrightarrow$ minimizing $S(A) = \int \frac{1}{2} \|dA\|^2 + J \wedge A$

E&M $\qquad A \in \Omega^1(\mathbb{R}^4).$

Example:

$C^\infty([a,b]) \ni q. \qquad q(a) = q_0, \quad q(b) = q_1$

$$S(q) = \int_a^b \left(\underbrace{\tfrac{1}{2} m \dot{q}^2}_{\text{kinetic energy}} - V(q)\right) dt$$

if x is an extreme $\Longrightarrow$ $\Delta f = f(x + \delta x) - f(x)$ is tiny when δx is tiny

$$\frac{\partial f(x + \varepsilon \delta x)}{\partial \varepsilon} = 0 \quad \text{at } \varepsilon = 0.$$

then $\dfrac{\partial S(q + \varepsilon \delta q)}{\partial \varepsilon} = 0 \quad$ at $\varepsilon = 0.$

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