Problem Set 18 — MAT257

March 29, 2017

Disclaimer—This page has been typeset by a student as a *convenient* consolidation of the homework problems. There inevitably will be mistakes; always defer to the official handout!

$*$ **Update 28/03/2017: Added the missing sign factor to ω in Problem A.**

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1 "Ponder..."

1. Exercises in Munkres §34.
2. Exercises in Munkres §35.

2 Solve and submit!

A. Consider S^{n-1} at the boundary of $D^n \subset \mathbb{R}^n$, taken with its standard orientation, and let $\iota : S^{n-1} \to \mathbb{R}^n$ be the inclusion map. Let

$$\omega = \iota^* \left(\sum_i (-1)^i x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \right) \in \Omega^{\text{top}}(S^{n-1}).$$

Prove that if $(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$ is a positively oriented basis of $T_x S^{n-1}$ for some $\mathbf{x} \in S^{n-1}$, then $\omega(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$ is the volume of the $(n-1)$-dimensional parallelepiped spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$, and hence for any smooth function f on S^{n-1}, $\int_{S^{n-1}} f\omega = \int_{S^{n-1}} f \mathrm{d}V$, where the latter integral is integration relative to the volume, as defined a long time ago.

B. *An alternative definition for "orientation".*

Define a **norientation** ("new orientation") of a vector space V to be a function

$$\nu : \{\text{ordered bases of } V\} \to \{\pm 1\}$$

which satisfies

$$\nu(v) = \operatorname{sign}\left(\det(C_v^u)\right) \nu(u)$$

whenever u and v are ordered bases of V and C_v^u is the change-of-basis matrix between them.

1. Explain how if $\dim(V) > 1$, a norientation is equivalent to an orientation.
   If $\dim(V) > 1$, then

2. Come up with a reasonable definition of a norientation of a k-dimensional manifold.
3. Explain how a norientation of M induces a norientation of ∂M.
4. What is a 0-dimensional manifold? What is a norientation of a 0-dimensional manifold?

   A 0-dimensional manifold is a set of discrete points. A norientation of a 0-dimensional manifold M is simply an ordering on the elements of M.
5. What is the integral of a 0-form on a 0-dimensional noriented manifold?
6. What is $\partial[0,1]$ as a noriented 0-manifold? (Assume that $[0,1]$ is endowed with its “positive” or “standard” orientation/norientation).

C. Let $\omega = y\,dx \in \Omega^1\left(\mathbb{R}^2_{x,y}\right)$.

1. Let Γ be the graph in $\mathbb{R}^2_{x,y}$ of some smooth function $f\colon [a,b] \to \mathbb{R}$. Using the inclusion of Γ to $\mathbb{R}^2_{x,y}$, consider ω also as a 1-form on Γ. What is $\int_\Gamma \omega$?
2. Prove that if E is an ellipse in $\mathbb{R}^2_{x,y}$ (of whatever major and minor axes, placed anywhere and tilted as you please), then $\int_{\partial E} \omega$ is the area of E.
3. Compute also $d\omega$ and $\int_E d\omega$.