Feb 15th Wed, hour 054
Read along: 27 - 29.
Today. Wedge ~~ten~~ tangents

Thm: $\exists ! \wedge : A^k(V) \wedge A^l(V) \to A^{k+l}(V)$ such that
1. $\wedge$ is associative, bilinear
2. $\wedge$ is skew-symmetric $F \in A^k, g \in A^l \Rightarrow F \wedge g = (-1)^{k \cdot l} g \wedge F$
3. $\psi_I = \phi_{i_1} \wedge \cdots \wedge \phi_{i_k}$, if $I = (i_1, \cdots, i_k) \in \binom{n}{k}$

} only need Uniqueness.

where $\phi_i(a_j) = \delta_{ij}$; $F = \sum \alpha_I \psi_I$, $g = \sum \beta_J \psi_J$

$(F \wedge g) = \sum_{I,J} \alpha_I \beta_J (\psi_I \wedge \psi_J)$, where

$\psi_I \wedge \psi_J \overset{3}{=} (\phi_{i_1} \wedge \phi_{i_2} \wedge \cdots \wedge \phi_{i_k}) \wedge (\phi_{j_1}, \cdots, \phi_{j_l})$

$\overset{1}{=} \phi_{i_1} \wedge \cdots \wedge \phi_{i_k} \wedge \phi_{j_1} \wedge \cdots \wedge \phi_{j_l}$

repeatedly use 2. $\overset{2}{=} \pm \phi_{q_1} \wedge \phi_{q_2} \wedge \cdots \wedge \phi_{q_{k+l}} = \begin{cases} \pm \phi_Q & \text{if } q_1 < q_2 < \cdots < q_{k+l}, \text{ while set } Q \text{ be } \{q_1, \cdots, q_{k+l}\} \in \binom{n}{k+l} \\ 0 & \text{if } q_4 = q_5 = 7 \wedge \phi_{q_4} \wedge \phi_{q_5} \wedge \cdots = \cdots \phi_7 \wedge \phi_7 \cdots \overset{1/2}{=} -(\cdots \phi_7 \wedge \phi_7 -) \end{cases}$

$k=2$ $k=3$

Ex1: $(\phi_2 \wedge \phi_5) \wedge (\phi_1 \wedge \phi_4 \wedge \phi_{257})$
$= (-1)^2 \phi_1 \wedge \phi_2 \wedge \phi_5 \wedge \phi_4 \wedge \phi_{257} = -\phi_1 \wedge \phi_2 \wedge \phi_4 \wedge \phi_5 \wedge \phi_{257}$

Ex2. $(\phi_2 \wedge \phi_5) \wedge (\phi_1 \wedge \phi_5 \wedge \phi_{257}) = 0$

$V \xrightarrow{T} W$

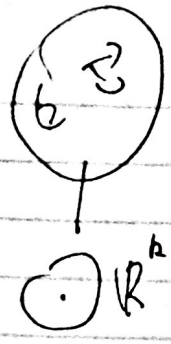
$A^k(V) \xleftarrow{T^*} A^k(W)$

$T^* F \leftarrow F$

$X_i \in V$,
$(T^* \in F)(X_1, \cdots, X_k) = F(TX_1, \cdots, TX_k)$
claim: $F \in A^k(W)$, $g \in A^l(W)$:
$(T^* F) \wedge T^*(g) = T^*(F \wedge g)$

$\underbrace{\underset{A^k(V)}{\rotatebox{90}{$\in$}} \quad \underset{A^l(V)}{\rotatebox{90}{$\in$}}}_{\in A^{k+l}(V)}$ $\qquad$ $\underset{A^{k+l}(W)}{\rotatebox{90}{$\in$}} \to A^{k+l}(V)$

LHS: Let $X_1, \cdots, X_{k+l} \in V$,
$(T^* F) \wedge (T^* g)(X_1, \cdots, X_{k+l})$
$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma (T^* F)(X_{\sigma 1}, \cdots, X_{\sigma k}) \cdot (T^* g)(X_{\sigma k+1}, \cdots, X_{\sigma k+l})$
$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma F(TX_{\sigma 1}, \cdots, TX_{\sigma k}) \cdot g(\cdots) = RHS$



tangent vector to $\mathbb{R}^k$

Def: a tangent vector $\vec{\xi}$ to $\mathbb{R}^n$, is a pair $\xi = (X, V)$ where $X, V \in \mathbb{R}^n$

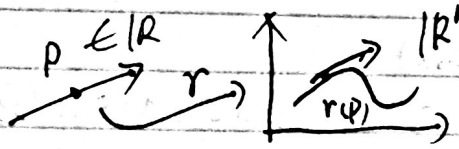
Def: a tangent vector ξ to $\mathbb{R}^n$ ~~is a pair~~ at X, is $T_X(\mathbb{R}^n) = \{(X, V) : V \in \mathbb{R}^n\}$. isomorphic $\simeq \mathbb{R}^n$

$T_X(\mathbb{R}^n)$ is a v.s. 1. $\xi_1 + \xi_2$. $\xi_i = (X, V_i) \in T_X(\mathbb{R}^n)$
$= (X, V_1) + (X, V_2) := (X, V_1 + V_2)$

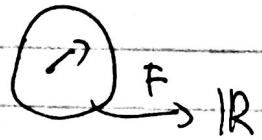
always with same X. 2. $\lambda(X, V) = (X, \lambda V)$ ...

3. $(X_1, V_1) + (X_2, V_2)$ is meaningless

Get tangent vectors:



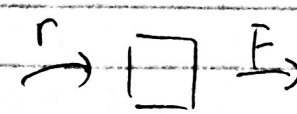
tangent to a curve $r: \mathbb{R} \to \mathbb{R}^n$ (C^1) at $r(p)$ is $(r(p), D_{r(p)}(e_1))$



Do with tangent vectors:

Def: If $\xi \in T_X(\mathbb{R}^n)$ and $F: U \to \mathbb{R}$, where U is a nbd of X, assume F is diffable, then directional derivative of F in the direction of ξ to be $D_\xi F = \lim_{h \to 0} \frac{F(x+hv) - F(x)}{h}$

$= D_{Fx} \cdot V$



, $\frac{d(F \cdot r)}{dt} = D_{(r(t), D_{r(t)}(e_1))} F$